HW #8 – Solutions

#10.2 (a) $A = \begin{bmatrix} -1 & 4 \\ 1 & -5 \end{bmatrix}$ so $A^{-1} = \frac{1}{5-4}\begin{bmatrix} -5 & -4 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & -4 \\ -1 & -1 \end{bmatrix}$

$\|A\|_1 = \max(2, 9) = 9$ $\qquad \|A^{-1}\|_1 = \max(6, 5) = 6$

$\|A\|_\infty = \max(5, 6) = 6$ $\qquad \|A^{-1}\|_\infty = \max(9, 2) = 9$

so $K_1(A) = \|A\|_1 \|A^{-1}\|_1 = 54$

& $K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 54$

(b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 2 & 5 & 1 \end{bmatrix}$ and (by MATLAB) $A^{-1} = \begin{bmatrix} 11 & -3 & -2 \\ -4 & 1 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

$\|A\|_1 = \max(5, 11, 5) = 11$ $\qquad \|A^{-1}\|_1 = \max(17, 5, 3) = 17$

$\|A\|_\infty = \max(4, 9, 8) = 9$ $\qquad \|A^{-1}\|_\infty = \max(16, 6, 3) = 16$

so $K_1(A) = \|A\|_1 \|A^{-1}\|_1 = 187$

& $K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 144$.

#10.3: Claim: $K(A) = K(A^{-1})$.

Proof: By definition, $K(A) = \|A\| \|A^{-1}\|$.

and $K(A^{-1}) = \|A^{-1}\| \|(A^{-1})^{-1}\| = \|A^{-1}\| \|A\| = K(A)$

$\uparrow$ because $(A^{-1})^{-1} = A$.

#10.6 If $A = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with each $d_i \neq 0$ (to ensure A is nonsingular)

then $A^{-1} = \mathrm{diag}\left(\frac{1}{d_1}, \frac{1}{d_2}, \ldots, \frac{1}{d_n}\right)$.

We know from Ch. 9 (#9.9) that $\|A\|_1 = \|A\|_2 = \|A\|_\infty = \max(|d_1|, |d_2|, \ldots, |d_n|)$.

But, by the same argument, $\|A^{-1}\|_1 = \|A^{-1}\|_2 = \|A^{-1}\|_\infty = \max\left(\frac{1}{|d_1|}, \frac{1}{|d_2|}, \ldots, \frac{1}{|d_n|}\right)$

$= 1/\min(|d_1|, |d_2|, \ldots, |d_n|)$.

Thus, $K_1(A) = K_2(A) = K_\infty(A) = \dfrac{\max(|d_1|, |d_2|, \ldots, |d_n|)}{\min(|d_1|, |d_2|, \ldots, |d_n|)}$.

#10.7 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x' = \begin{bmatrix} -1.3 \\ 1.2 \end{bmatrix}$.

$K_\infty(A) = 21$ (from #10.1 — done in class)

$b' = Ax' = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix}$, $r = b - b' = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$, so $K_\infty(A) \dfrac{\|b-b'\|_\infty}{\|b\|_\infty} = 21 \dfrac{0.1}{1} = 2.1$

$x = A^{-1}b = \dfrac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \dfrac{-1}{2}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so $e = x - x' = \begin{bmatrix} 0.3 \\ -0.2 \end{bmatrix}$ and $\dfrac{\|x-x'\|_\infty}{\|x\|_\infty} = \dfrac{0.3}{1} = 0.3$

Because $0.3 = \dfrac{\|x-x'\|_\infty}{\|x\|_\infty} \le K_\infty(A) \dfrac{\|b-b'\|_\infty}{\|b\|_\infty} = 2.1$, we see that (10.20) holds.

#10.9 $A = \begin{bmatrix} 0.999 & 1 \\ 1 & 0.999 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $x' = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

$b' = Ax' = \begin{bmatrix} 1.998 \\ 2 \end{bmatrix}$ so the residual vector is $r = b - b' = \begin{bmatrix} 0.002 \\ 0 \end{bmatrix}$.

The residual equation is $Ae = r$.

#10.10. (#10.9 cont.) The exact solution to $Ax = b$ is $x = A^{-1}b = \dfrac{1}{0.999^2 - 1}\begin{bmatrix} 0.999 & -1 \\ -1 & 0.999 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} \frac{2 - 1.998}{1 - 0.999^2} \\ \frac{-2 + 1.998}{1 - 0.999^2} \end{bmatrix} = \begin{bmatrix} 1.0005002 \\ 1.0005002 \end{bmatrix}$

$e = x - x' = \begin{bmatrix} -0.9994997 \\ 1.0005002 \end{bmatrix}$ so $\dfrac{\|x-x'\|_\infty}{\|x\|_\infty} = \dfrac{1.0005002}{1.0005002} = 1.$ ← relative error of x'

while $\dfrac{\|b-b'\|_\infty}{\|b\|_\infty} = \dfrac{0.002}{2} = 0.001.$ ← relative error of RHS

#10.11 (#10.9 cont.) $K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = (1.999)(999.999\ldots) = 1999$ ← fairly large!

This large value for $K_\infty(A)$ is needed to ensure (10.20) holds:

$\dfrac{\|x-x'\|_\infty}{\|x\|_\infty} = 1 \le 1999(0.001) = K_\infty(A)\dfrac{\|b-b'\|_\infty}{\|b\|_\infty} = 1999(0.001) = 1.999.$

#10.13. $x = 0.0001352$, $x' = 0.0001376$

$= 1.352 \times 10^{-4}$ $(p = -4)$

$|x - x'| = |-0.0000024| = 2.4 \times 10^{-6}$ so $\dfrac{|x-x'|}{|x|} = \dfrac{2.4\times 10^{-6}}{1.352 \times 10^{-4}} = 0.01775$

$= 0.1775 \times 10^{-1}$

$= 1.775 \times 10^{-2}$

this is the more usual answer.

This means x' has between 1 & 2 significant digits as an approximation to x.

#10.15. From Exercise #9.12: $\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{n}\|A\|_\infty$.

Applying the same result to A^{-1}: $\frac{1}{\sqrt{n}}\|A^{-1}\|_\infty \le \|A^{-1}\|_2 \le \sqrt{n}\|A^{-1}\|_\infty$.

Thus. $\frac{1}{\sqrt{n}^2}\|A\|_\infty\|A^{-1}\|_\infty \le \|A\|_2\|A^{-1}\|_2 \le \sqrt{n}^2\|A\|_\infty\|A^{-1}\|_\infty$

or $\frac{1}{n}K_\infty(A) \le K_2(A) \le n K_\infty(A).$