HW 7 - Solutions

#9.1. $x = \begin{bmatrix} -5 \\ 4 \\ -3 \\ 6 \end{bmatrix}$.

$$\|x\|_1 = |-5| + |4| + |-3| + |6| = 5+4+3+6 = 18$$

$$\|x\|_2 = \left((-5)^2 + (4)^2 + (-3)^2 + (6)^2\right)^{1/2} = 86^{1/2}$$

$$\|x\|_\infty = \max\left(|-5|, |4|, |-3|, |6|\right) = 6.$$

#9.4. Claim: $\|I\| = 1$.

Proof: $\|I\| = \max_{x \neq 0} \dfrac{\|Ix\|}{\|x\|} = \max_{x\neq 0} \dfrac{\|x\|}{\|x\|} = \max 1 = 1.$ $\boxed{//}$

by def'n of matrix norm.

because $Ix = x$ for all x

because $\|x\| \neq 0$ for all $x \neq 0$.

#9.5. $A = \begin{bmatrix} -5 & 4 & 2 \\ 3 & -6 & -9 \\ 3 & 7 & -1 \end{bmatrix}$

$$\begin{aligned} \|A\|_1 &= \max\left(|5|+|3|+|3|, |4|+|-6|+|7|, |2|+|-9|+|-1|\right) \\ &= \max(11, 17, 12) \\ &= 17 \end{aligned}$$

$$\begin{aligned} \|A\|_\infty &= \max\left(|-5|+|4|+|2|, |3|+|-6|+|-9|, |3|+|7|+|-1|\right) \\ &= \max(11, 18, 13) \\ &= 18. \end{aligned}$$

#9.6. Claim: Let $\|\cdot\|$ be a norm for $\mathbb{R}^n$. Then $\big|\, \|x\| - \|y\| \,\big| \leq \|x-y\|$.

Proof: Notice that $\|x\| = \|x - y + y\| = \|(x-y) + y\| \leq \|x-y\| + \|y\|$

and $\|y\| = \|y - x + x\| = \|(y-x) + x\| \leq \|y-x\| + \|x\|$.

Because $\|x-y\| = \|y-x\|$ we see that $\|x-y\| \geq \|x\| - \|y\|$

and $\|x-y\| \geq \|y\| - \|x\|$.

we can conclude that

$$\|x-y\| \geq \max\left(\|x\| - \|y\|, \|y\| - \|x\|\right) = \big|\, \|x\| - \|y\| \,\big| \quad \boxed{\Xi}$$

#9.7. $\|A\|_1$ is the maximum absolute column sum of A.

$\|A^t\|_\infty$ is the maximum absolute row sum of A^t.

But, because the rows of A^t are the columns of A, $\|A\|_1 = \|A^t\|_\infty$.

#9.12 Let A be an $n \times n$ matrix.

Recall that the vector norms satisfy:

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty.$$

and that the matrix norms are defined by

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}.$$

Putting all of this together yields:

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$ def. of matrix 2-norm.

$$\le \max_{x\neq 0} \frac{\sqrt{n}\,\|Ax\|_\infty}{\|x\|_2}$$ because $\|Ax\|_2 \le \sqrt{n}\,\|Ax\|_\infty$

$$\le \max_{x\neq 0} \frac{\sqrt{n}\,\|Ax\|_\infty}{\|x\|_\infty}$$ because $0 < \|x\|_\infty \le \|x\|_2$. (for $x \neq 0$)

$$= \sqrt{n} \max \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

$$= \sqrt{n}\,\|A\|_\infty.$$

and

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$ def. of matrix 2-norm

$$\ge \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_2}$$ because $\|Ax\|_2 \ge \|Ax\|_\infty$

$$\ge \max_{x\neq 0} \frac{\|Ax\|_\infty}{\sqrt{n}\,\|x\|_\infty}$$ because $\|x\|_2 \le \sqrt{n}\,\|x\|_\infty$.

$$= \frac{1}{\sqrt{n}} \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

$$= \frac{1}{\sqrt{n}}\,\|A\|_\infty.$$

When these two statements are ~~defined~~ combined:

$$\frac{1}{\sqrt{n}}\,\|A\|_\infty \le \|A\|_2 \le \sqrt{n}\,\|A\|_\infty$$