# HW #6 - Solutions

#7-1: $A = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 0 & -1 & 5 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}, \ b = \begin{bmatrix} -4 \\ 3 \\ -3 \\ -4 \end{bmatrix}$

Because $A$ is tridiagonal, we can find $L$ & $U$ by overwriting $A$ as follows:

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ \cancel{-2}\,-1 & \cancel{1}\,-1 & 1 & 0 \\ 0 & \cancel{-1}\,1 & \cancel{5}\,4 & 2 \\ 0 & 0 & \cancel{2}\,\tfrac{1}{2} & \cancel{2}\,1 \end{bmatrix} \text{ so that } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \tfrac{1}{2} & 1 \end{bmatrix}, \ U = \begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: Check that $LU = A$.

To solve $Ax=b$ we solve $Ly=b$ (overwriting $b$) & then $Ux=y$ (overwriting $b$ again)

$$b = \begin{bmatrix} -4 \\ 3 \\ -3 \\ -4 \end{bmatrix} : \ y = \begin{bmatrix} -4 \\ 3+(-4) \\ -3-(-1) \\ -4-\tfrac{1}{2}(-2) \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -2 \\ -3 \end{bmatrix} \quad \text{(Check that } Ly=b\text{)}$$

$$x = \begin{bmatrix} (-4+2(2))/2 \\ (-1-1)/-1 \\ (-2-2(-3))/4 \\ -3/1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -3 \end{bmatrix} \quad \text{(Check that } Ux=y \text{ \& } Ax=b\text{)}$$

#7-2: Same approach for #7-1:

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ \cancel{-2}\,1 & \cancel{4}\,3 & 6 & 0 \\ 0 & \cancel{2}\,\tfrac{2}{3} & \cancel{5}\,1 & -1 \\ 0 & 0 & \cancel{-1}\,-1 & \cancel{3}\,2 \end{bmatrix} \text{ so that } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \tfrac{2}{3} & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \ \& \ U = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} -2 \\ -2 \\ -5 \\ 17 \end{bmatrix} : \ y = \begin{bmatrix} -2 \\ -2-(-2) \\ -5-\tfrac{2}{3}(0) \\ 17+1(-5) \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -5 \\ 12 \end{bmatrix}; \ x = \begin{bmatrix} (-2-(-2))/(-2) \\ (0-6(1))/3 \\ (-5+6)/1 \\ 12/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 6 \end{bmatrix}$$

#7-3: Performing the tridiagonal L U factorization:

$$\begin{bmatrix} 2 & -1 & & & \\ \cancel{-1}\,\tfrac{1}{2} & \cancel{2}\,\tfrac{3}{2} & -1 & & \\ & \cancel{-1}\,-\tfrac{2}{3} & \cancel{2}\,\tfrac{4}{3} & -1 & \\ & & \cancel{-1}\,-\tfrac{3}{4} & \cancel{2}\,\tfrac{5}{4} & -1 \\ & & & \cancel{-1}\,-\tfrac{4}{5} & \cancel{2}\,\tfrac{6}{5} \end{bmatrix}$$

so $L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \tfrac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & -\tfrac{2}{3} & 1 & 0 & 0 \\ 0 & 0 & -\tfrac{3}{4} & 1 & 0 \\ 0 & 0 & 0 & -\tfrac{4}{5} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 0 & 5/4 & -1 \\ 0 & 0 & 0 & 0 & 6/5 \end{bmatrix}$

In general: $l_k = -\dfrac{k}{k+1}$, $k = 1, 2, \dots, n-1$ and $d_k = \dfrac{k+1}{k}$, $k = 1, 2, \dots, n$

& $u_k = -1$ $(k = 1, 2, \dots, n-1)$.

#7-4: (a) $|-3| \geq |1| + |-1| + |0|$

$|-5| \geq |2| + |-1| + |-2|$

$|6| \geq |1| + |-1| + |3|$

$|-4| \geq |0| + |2| + |-2|$

$\therefore$ this matrix

$$\begin{bmatrix} -3 & 2 & 1 & 0 \\ 1 & -5 & -1 & 2 \\ -1 & -1 & 6 & -2 \\ 0 & -2 & 3 & -4 \end{bmatrix}$$ is column diagonally dominant

(b) column 3: $|3| < |0| + |2| + |-2|$

$\therefore \begin{bmatrix} 3 & -2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & -1 & 3 & -3 \\ 0 & 0 & -2 & 4 \end{bmatrix}$ is not column diagonally dominant

(c) $\begin{bmatrix} -2 & -1 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -1 & -3 \end{bmatrix}$ is column diagonally dominant

#7-5: (a) does not satisfy Theorem 7.8 because the matrix is <u>not</u> tridiagonal.

(b) does not satisfy Theorem 7.8 because the matrix is not column diagonally dominant.

(c) does satisfy the conditions in Theorem 7.8

$\left\{ \begin{array}{l} \text{upper diagonal comp. } \neq 0 \\ \text{column diag. dominant} \\ |a_{11}| = 2 > 1 = |a_{21}| \end{array} \right\}$.

#8-2: $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 0 & 8 \\ -1 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} = uv^t$ so $u = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ & $v = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ 3.

$1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad 0 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad 4 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Note: There are many other choices for u & v that will work.

#8-3: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix}$ since column 3 is not a multiple of the first two columns, this matrix cannot be written as an outer product.

$1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad * \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

must be a −

#8-4: $A = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix} = I - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & -1 \end{bmatrix}$

$= B - uv^t$ with $B = I_{5\times5}$, $u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ (this answer is not unique).

#8.5. Step 1: Solve $Bz = u$: $z = u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Step 2: Compute $\gamma = 1 - v^t z = 1 - (-1+0+0+0-1) = 3 \neq 0$ (nonsingular!)

Step 3: Solve $By = b$: $y = b = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 3 \\ 3 \end{bmatrix}$

Step 4: Compute $\beta = \dfrac{v^t y}{\gamma} = \dfrac{(-3+0+0+0-3)}{3} = -2$

Step 5: Compute $x = y + \beta z = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 3 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

Check: $Ax = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 3 \\ 3 \end{bmatrix}$

#8.6 $A = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 1 \end{bmatrix}$ & $b = \begin{bmatrix} -7 \\ -1 \\ 4 \\ 9 \end{bmatrix}$.

The challenge here is to pick an appropriate B that is nonsingular & is easy to "solve". Our first choice would be to pick B as a diagonal matrix but this would not fit our form: $A = B - uv^t$. Our next choice would be a tridiagonal system but this matrix A has only one non-zero component above the main diagonal. This would seem to be more appropriate for our next choice: a triangular matrix.

Let $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 1 \end{bmatrix}$. It will be easy to solve systems involving B. (forward substitution only; no factorization)

Now $-uv^t = A - B = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = -\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$. so $u = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ & $v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Now, applying the solution algorithm:

Step 1: Solve $Bz = u$: $z = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -12 \end{bmatrix}$ $Bz = u$ ✓

Step 2: Compute $\gamma = 1 - v^t z = 1 - (-12) = 13 \neq 0$ so A is nonsingular.

Step 3: Solve $By = b$: $y = \begin{bmatrix} -7 \\ -6 \\ 11 \\ 52 \end{bmatrix}$ $By = b$ ✓

Step 4: Compute $\beta = \dfrac{v^t y}{\gamma} = \dfrac{52}{13} = 4$

Step 5: Compute $x = y + \beta z = \begin{bmatrix} -7 \\ -6 \\ 11 \\ 52 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 2 \\ -2 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $Ax = b$ ✓

#8.8: $A = \begin{bmatrix} 1 & 2 & -1 & -1 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} -4 \\ 2 \\ 4 \\ 3 \end{bmatrix}$

$$-uv^t = A - B = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \text{ so } u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Step 1: Solve $Bz = u$ (using L-U factorization)

$Ly = u$ : $y = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$; $Uz = y$ : $z = \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ $\quad Bz = u$ ✓

Step 2: Compute $\gamma = 1 - v^t z = 1 - (1(-1)) = 2 \neq 0$ (A,B nonsingular)

Step 3: Solve $By = b$ (using L-U factorization)

$Lw = b$ : $w = \begin{bmatrix} -4 \\ -2 \\ 6 \\ 1 \end{bmatrix}$, $Uy = w$ : $y = \begin{bmatrix} 7 \\ -5 \\ 4 \\ 1 \end{bmatrix}$ $\quad By = b$ ✓

Step 4: Compute $\beta = \dfrac{v^t y}{\gamma} = \dfrac{4}{2} = 2$

Step 5: Compute $x = y + \beta z = \begin{bmatrix} 7 \\ -5 \\ 4 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} +1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ $\quad Ax = b$ ✓