HW#5 – Solutions

#6-1a. $\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \xrightarrow{②-2①\to②} \begin{bmatrix} 1 & 4 \\ 0 & \boxed{0} \end{bmatrix}$ $\therefore$ this matrix, $\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$, is singular.

c. $\begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & -1 \\ 5 & -7 & -2 \end{bmatrix} \xrightarrow[③-5①\to③]{②-2①\to②} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{③-2②\to③} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & \boxed{0} \end{bmatrix}$ $\therefore$ $\begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & -1 \\ 5 & -7 & -2 \end{bmatrix}$ is singular

#6-2a. $A = \begin{bmatrix} -1 & 4 \\ 1 & -5 \end{bmatrix}$. To begin, find the LU factorization of A

$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -5 \end{bmatrix}$

$\xrightarrow{②+①\to②}$

$\begin{bmatrix} 1 & \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -1 \end{bmatrix}$ so A is invertible & $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -1 \end{bmatrix} = LU$.

Now to find A^{-1} we solve $Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $Ax = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$: $Ly = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Ux = y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

$Ax = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $Ly = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so $A^{-1} = \begin{bmatrix} -5 & -4 \\ -1 & -1 \end{bmatrix}$

$Ux = y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$

Check: $AA^{-1} = \begin{bmatrix} -1 & 4 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -5 & -4 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

#6-2b. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$. First, write A = LU.

$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$

$\xrightarrow{③-①\to③}$

$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

$\xrightarrow{③-2②\to③}$

$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ so $A = LU$ with $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

(& A is invertible).

Now, solve $Ax=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $Ax=\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $Ax=\begin{bmatrix}0\\0\\1\end{bmatrix}$:

$Ax=\begin{bmatrix}1\\0\\0\end{bmatrix}$: $Ly=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\ & & \end{bmatrix}y=\begin{bmatrix}1\\0\\0\end{bmatrix} \Longrightarrow y=\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$Ux=y$, $\begin{bmatrix}1&1&0\\0&1&1\\0&0&-1\end{bmatrix}x=\begin{bmatrix}1\\0\\-1\end{bmatrix} \Longrightarrow x=\begin{bmatrix}2\\-1\\1\end{bmatrix}$

$Ax=\begin{bmatrix}0\\1\\0\end{bmatrix}$: $Ly=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\1&2&1\end{bmatrix}y=\begin{bmatrix}0\\1\\0\end{bmatrix} \Longrightarrow y=\begin{bmatrix}0\\1\\-2\end{bmatrix}$

$Ux=y$, $\begin{bmatrix}1&1&0\\0&1&1\\0&0&-1\end{bmatrix}x=\begin{bmatrix}0\\1\\-2\end{bmatrix} \Longrightarrow x=\begin{bmatrix}1\\-1\\2\end{bmatrix}$

$Ax=\begin{bmatrix}0\\0\\1\end{bmatrix}$: $Ly=\begin{bmatrix}0\\0\\1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\1&2&1\end{bmatrix}y=\begin{bmatrix}0\\0\\1\end{bmatrix} \Longrightarrow y=\begin{bmatrix}0\\0\\1\end{bmatrix}$

$Ux=y$, $\begin{bmatrix}1&1&0\\0&1&1\\0&0&-1\end{bmatrix}x=\begin{bmatrix}0\\0\\1\end{bmatrix} \Longrightarrow x=\begin{bmatrix}-1\\1\\-1\end{bmatrix}$

Thus, $A^{-1}=\begin{bmatrix}2&1&-1\\-1&-1&1\\1&2&-1\end{bmatrix}$

Check: $AA^{-1}=\begin{bmatrix}1&1&0\\0&1&1\\1&3&1\end{bmatrix}\begin{bmatrix}2&1&-1\\-1&-1&1\\1&2&-1\end{bmatrix}$

$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ ✓

#6-3a. $-x_1+4x_2=3$
$x_1-5x_2=0$

$\begin{bmatrix}-1&4\\1&-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}$

$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-1&4\\1&-5\end{bmatrix}^{-1}\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}-5&-4\\-1&-1\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}-15\\-3\end{bmatrix}.$

#6-3b. $x_1+x_2=-1$
$x_2+x_3=2$
$x_1+3x_2+x_3=3$

$\begin{bmatrix}1&1&0\\0&1&1\\1&3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1&1&0\\0&1&1\\1&3&1\end{bmatrix}^{-1}\begin{bmatrix}-1\\2\\3\end{bmatrix}$

$=\begin{bmatrix}2&1&-1\\-1&-1&1\\1&2&-1\end{bmatrix}\begin{bmatrix}-1\\2\\3\end{bmatrix}=\begin{bmatrix}-3\\2\\0\end{bmatrix}$

#6-4 Let's find the LU factorization of $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$.

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

$② - \frac{c}{a}① \to ②$

$$\begin{bmatrix} & \\ \frac{c}{a} & \end{bmatrix}\begin{bmatrix}a&b\\0&d-\frac{c}{a}b\end{bmatrix}$$

So $A=\begin{bmatrix}1&0\\ \frac{c}{a}&1\end{bmatrix}\begin{bmatrix}a&b\\0&d-\frac{c}{a}b\end{bmatrix}=LU.$

To know A is invertible we must have $a\neq 0$ & $d-\frac{c}{a}b\neq 0$.

If a common denominator is found: $d-\frac{c}{a}b=\frac{da-cb}{a}\neq 0$ means $da-cb\neq 0$.

Thus, A is invertible if and only if $a\neq 0$ and $ad-bc\neq 0$.

To find A^{-1} solve $Ax=\begin{bmatrix}1\\0\end{bmatrix}$, $Ax=\begin{bmatrix}0\\1\end{bmatrix}$.

$Ax=\begin{bmatrix}1\\0\end{bmatrix}$: $Ly=\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}1&0\\ \frac{c}{a}&1\end{bmatrix}y=\begin{bmatrix}1\\0\end{bmatrix}\Rightarrow y=\begin{bmatrix}1\\ -\frac{c}{a}\end{bmatrix}$

$Ux=y$, $\begin{bmatrix}a&b\\0&\frac{ad-bc}{a}\end{bmatrix}x=\begin{bmatrix}1\\ -\frac{c}{a}\end{bmatrix}\Rightarrow x=\begin{bmatrix}\frac{d}{ad-bc}\\ \frac{-c}{ad-bc}\end{bmatrix}$

$Ax=\begin{bmatrix}0\\1\end{bmatrix}$: $Ly=\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}1&0\\ \frac{c}{a}&1\end{bmatrix}y=\begin{bmatrix}0\\1\end{bmatrix}\Rightarrow y=\begin{bmatrix}0\\1\end{bmatrix}$

$Ux=y$, $\begin{bmatrix}a&b\\0&\frac{ad-bc}{a}\end{bmatrix}x=\begin{bmatrix}0\\1\end{bmatrix}\Rightarrow x=\begin{bmatrix}\frac{-b}{ad-bc}\\ \frac{a}{ad-bc}\end{bmatrix}$

so $A^{-1}=\begin{bmatrix}\frac{d}{ad-bc}&\frac{-b}{ad-bc}\\ \frac{-c}{ad-bc}&\frac{a}{ad-bc}\end{bmatrix}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

Or, just show that the given formula for A^{-1} satisfies $AA^{-1}=A^{-1}A=I$

#6-6: Because A is nonsingular we know A is invertible. The inverse of A, A^{-1}, satisfies:
$AA^{-1}=I$ and $A^{-1}A=I$. Take the transpose of each equation:

1. $(AA^{-1})^t=I^t\Leftrightarrow(A^{-1})^tA^t=I$ and 2. $(A^{-1}A)^t=I^t\Leftrightarrow A^t(A^{-1})^t=I$.

We know A^t is also nonsingular. It's inverse, $(A^t)^{-1}$, satisfies:
$A^t(A^t)^{-1}=I$ and $(A^t)^{-1}A^t=I$. These are exactly the same equations found above. By uniqueness of the inverse of $(A^t)^{-1}$ we conclude $(A^t)^{-1}=(A^{-1})^t$.