HW #4 – Solutions

#5.3 In (5.6) we are told that $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 5 & -2 \\ 0 & 3 & 5 & 1 \end{bmatrix}$ can be written as $PA = LU$ for

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With this factorization we can solve $Ax = b$, with $b = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 9 \end{bmatrix}$ as follows:

1. $c = Pb = \begin{bmatrix} 4 \\ 2 \\ 9 \\ 3 \end{bmatrix}$

2. Forward substitution $Ly = c$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix} y = \begin{bmatrix} 4 \\ 2 \\ 9 \\ 3 \end{bmatrix}$

$y_1 = 4$

$y_2 = 2$

$3y_2 + y_3 = 9 \Rightarrow y_3 = 9 - 6 = 3$

$-y_1 + 3y_2 + y_4 = 3 \Rightarrow y_4 = 3 + 4 - 6 = 1.$

So $y = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix}$.

3. Backward substitution: $Ux = y$: $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix}$

$x_4 = 1$

$-x_3 + 4x_4 = 3 \Rightarrow x_3 = 4 - 3 = 1$

$x_2 + 2x_3 - x_4 = 2 \Rightarrow x_2 = 2 - 2 + 1 = 1$

$x_1 + 2x_2 + x_3 = 4 \Rightarrow x_1 = 4 - 2 - 1 = 1$

So $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ (this checks)

#5.4 (a)

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{matrix} ① \to ② \\ ② \to ① \end{matrix}$$

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad ② - \tfrac{1}{3}① \to ②$$

$$\begin{bmatrix} & \\ \tfrac{1}{3} & \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & \tfrac{2}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so $L = \begin{bmatrix} 1 & 0 \\ \tfrac{1}{3} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & 4 \\ 0 & \tfrac{2}{3} \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Note: Components below the main diagonal of L are all ≤ 1 in absolute value.

Check: $PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$LU = \begin{bmatrix} 1 & 0 \\ \tfrac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & \tfrac{2}{3} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ ✓

(b)

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ -2 & 4 & -8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ -2 & 4 & -8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{matrix} ③ \to ① \\ ① \to ③ \end{matrix}$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} -2 & 4 & -8 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} ② - \tfrac{1}{2}① \to ② \\ ③ + \tfrac{1}{2}① \to ③ \end{matrix}$$

$$\begin{bmatrix} & & \\ \tfrac{1}{2} & & \\ -\tfrac{1}{2} & & \end{bmatrix} \begin{bmatrix} -2 & 4 & -8 \\ 0 & -1 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} ③ \to ② \\ ② \to ③ \end{matrix}$$

$$\begin{bmatrix} & & \\ -\tfrac{1}{2} & & \\ \tfrac{1}{2} & & \end{bmatrix} \begin{bmatrix} -2 & 4 & -8 \\ 0 & 3 & -3 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad ③ + \tfrac{1}{3}② \to ③$$

$$\begin{bmatrix} & & \\ -\tfrac{1}{2} & & \\ \tfrac{1}{2} & -\tfrac{1}{3} & \end{bmatrix} \begin{bmatrix} -2 & 4 & -8 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

so $L = \begin{bmatrix} 1 & 0 & 0 \\ -\tfrac{1}{2} & 1 & 0 \\ \tfrac{1}{2} & -\tfrac{1}{3} & 1 \end{bmatrix}$, $U = \begin{bmatrix} -2 & 4 & -8 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Note: $LU = \begin{bmatrix} -2 & 4 & -8 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = PA$. (& this checks)

#5.5. (a) $x_1+2x_2=1$, $3x_1+4x_2=1$ $\iff Ax=b$ where $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$, $b=\begin{bmatrix}1\\1\end{bmatrix}$.

From #5.4 (a) we know $PA=LU$ with $P=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $L=\begin{bmatrix}1&0\\1/3&1\end{bmatrix}$, $U=\begin{bmatrix}3&4\\0&2/3\end{bmatrix}$.

So: 1) $c=Pb=\begin{bmatrix}1\\1\end{bmatrix}$

2) Forward subst: $Ly=c$: $\begin{bmatrix}1&0\\1/3&1\end{bmatrix}y=\begin{bmatrix}1\\1\end{bmatrix} \Rightarrow y=\begin{bmatrix}1\\2/3\end{bmatrix}$

3) Backward subst: $Ux=y$: $\begin{bmatrix}3&4\\0&2/3\end{bmatrix}x=\begin{bmatrix}1\\2/3\end{bmatrix} \Rightarrow x=\begin{bmatrix}-1\\1\end{bmatrix}$

Check: $Ax=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}=b.$

(b) $x_1+x_2+x_3=2$, $-x_1+x_2-x_3=-2$, $-2x_1+4x_2-8x_3=-10$ $\iff Ax=b$ where $A=\begin{bmatrix}1&1&1\\-1&1&-1\\-2&4&-8\end{bmatrix}$, $b=\begin{bmatrix}2\\-2\\-10\end{bmatrix}$.

From #5.4 (b) we know $PA=LU$ with $P=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$, $L=\begin{bmatrix}1&0&0\\-1/2&1&0\\1/2&-1/3&1\end{bmatrix}$, $U=\begin{bmatrix}-2&4&-8\\0&3&-3\\0&0&2\end{bmatrix}$

So: 1) $c=Pb=\begin{bmatrix}-10\\2\\-2\end{bmatrix}$

2) Forward subst: $Ly=c$: $\begin{bmatrix}1&0&0\\-1/2&1&0\\1/2&-1/3&1\end{bmatrix}y=\begin{bmatrix}-10\\2\\-2\end{bmatrix} \Rightarrow y=\begin{bmatrix}-10\\-3\\2\end{bmatrix}$

3) Backward subst: $Ux=y$: $\begin{bmatrix}-2&4&-8\\0&3&-3\\0&0&2\end{bmatrix}x=\begin{bmatrix}-10\\-3\\2\end{bmatrix} \Rightarrow x=\begin{bmatrix}1\\0\\1\end{bmatrix}$

Check: $Ax=\begin{bmatrix}1&1&1\\-1&1&-1\\-2&4&-8\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}2\\-2\\-10\end{bmatrix}=b.$

#5.6 To solve $Ax=b$ by the L-U-P factorization of A we must first find L, U, & P. Recall (see pp. 7-8 in Ch. 3) that the LU factorization requires $\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n$ flops.

The only additional computations required to find the L-U-P factorization are the comparisons needed to identify the largest component (in abs. value) in a column (below the main diagonal). [We assume that row interchanges take no flops. A comparison typically requires a subtraction - 1 flop.].

To find the largest component in column 1 requires $n-1$ comparisons.
———— " ———— 2 — " — $n-2$ — " —
———— " ———— 3 — " — $n-3$ — " —
———— " ———— k — " — $n-k$ — " —
⋮
———— " ———— $n-1$ — " — 1 — " —
———— " ———— n — " — 0 — " —

The total # of comparisons is $\sum_{k=1}^{n}(n-k) = \sum_{k=0}^{n-1} k = \frac{(n-1)n}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$

Now for the three steps in the solution process.

1. It would be foolish to compute $c = Pb$ by matrix multiplication $((n\times n)(n\times 1))$ would require $n\cdot 1\cdot(2n-1) = 2n^2 - n$ flops. A better way to compute Pb is to exploit the permutation structure of P. Each component of c can be found by (at most) n comparisons - looking for the single 1 in the row. Once a 1 is found in a row, no further comparisons are needed. Because each row of $I_{n\times n}$ appears exactly once in P, the total # of comparisons needed to find $c = Pb$ is $1+2+3+\dots+n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$. — a factor of 4 better than using matrix mult.

2 & 3. The forward substitution was found to take n^2 flops (see p. 3-7)
Backward substitution also requires n^2 flops.
(You can save n flops in the forward substitution because no division is needed)

In total, the flops count to solve $Ax=b$ by partial pivoting is:

$$\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) + \left(\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n\right) + \left(\frac{n^2}{2} + \frac{n}{2}\right) + \left(n^2 - n\right) + n^2 = \frac{2}{3}n^3 + \frac{5}{2}n^2 - \frac{7}{6}n \text{ flops.}$$