HW#3 – Solutions

Chapter 3

#3.4 Let A & B be two $n\times n$ matrices. Let $C = AB$.

Each component in C is of the form

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}}_{n \text{ products } \& \ (n-1) \text{ sums.}}$$

which requires $2n-1$ flops. Each of the n^2 ~~entries~~ components in C has the same complexity. The total flop count is $n^2(2n-1)$.

$$= 2n^3 - n^2.$$

#3.5. Let L be a lower triangular $n\times n$ matrix and x an n vector (column).

Then

$$Lx = \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & & \vdots \\ * & * & * & & \\ \vdots & & & \ddots & 0 \\ * & & \cdots & & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} * x_1 \\ * x_1 + * x_2 \\ * x_1 + * x_2 + * x_3 \\ \vdots \end{bmatrix}$$

	# mult	# sum
	1	0
	2	1
	3	2
	$\vdots$	
	n	$n-1$
total	$\frac{n(n+1)}{2}$	$\frac{(n-1)n}{2}$.

The total flop count here will be $\frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{n}{2}(n+1+n-1) =$

$$= \frac{n}{2}(2n) = n^2.$$

#3.6 Let u, v, w be column n-vectors.

uv^t is the $n\times n$ matrix whose $(i,j)^{th}$ component is $u_i v_j$ (1 flop to compute)

Since there are n^2 terms in this product, it requires $n^2(1) = n^2$ flops to find uv^t.

Then $(uv^t)w$ is the product of an $n\times n$ matrix & a column n-vector.

Each ~~the~~ term in $(uv^t)w$ is the product of row i of (uv^t) with ~~the~~ w.

~~This~~ ~~also~~ This requires n products & $(n-1)$ sums; so $2n-1$ flops per component.

There are a total of n components, so $n(2n-1)$ flops for the matrix-vector product. The total flops to compute $uv^t w = (uv^t)w$ is

$$n^2 + n(2n-1) = 3n^2 - n.$$

Alternately, $v^t w$ is a 1×1 matrix (i.e. a scalar) whose sole component is

$$\underbrace{v_1 w_1 + v_2 w_2 + \dots + v_n w_n}_{n \text{ products } + \ (n-1) \text{ additions.}} \qquad \text{Total flops: } 2n-1.$$

Then $u(v^t w)$ is really a scalar multiplication computation:
The i^{th} component of $u(v^t w)$ is $\underbrace{(v^t w)}_{\text{scalar}} u_i$ — 1 multiplication (1 flop).

Since there are n components, the total flop count for this product is n.
The total flops required to compute $uv^t w = u(v^t w)$ is:

$$2n-1+n = 3n-1.$$

This is exactly one order of magnitude less than the # of flops needed to compute $(uv^t)w$. It is more efficient to compute $\cancel{(uv^t)w}$ $u(v^t w)$.

#3.8 To show $\sum_{k=1}^{n} k^m = \Theta(n^{m+1})$ requires demonstration that
$\sum_{k=1}^{n} k^m \le C n^{m+1}$ for some C that does not depend on n.
Because $k \le n$ it is true that $k^m \le n^m$ for each $k = 1, \dots, n$.
Thus $\sum_{k=1}^{n} k^m = 1^m + 2^m + 3^m + \dots + n^m \le \underbrace{n^m + n^m + n^m + \dots + n^m}_{n \text{ terms}} = n(n^m) = n^{m+1}$.

(i.e. we can choose $C = 1$, if desired).

$\cancel{\#4.1.\ 3\begin{bmatrix}1\\-2\\3\end{bmatrix} - 2\begin{bmatrix}-1\\5\\1\end{bmatrix} = \begin{bmatrix}5\\-16\\11\end{bmatrix} =}$

#4.2. $2[1\ 0\ 3\ 2] + [3\ 1\ 0\ 5] = [5\ 1\ 6\ 9]$
$[1\ 0\ 3\ 2] - [3\ 1\ 0\ 5] = [-2\ -1\ 3\ -3]$
$3[1\ 0\ 3\ 2] - 2[3\ 1\ 0\ 5] = [-3\ -2\ 9\ -4]$

can be written as

$$\begin{bmatrix}2 & 1\\ 1 & -1\\ 3 & -2\end{bmatrix}\begin{bmatrix}1 & 0 & 3 & 2\\ 3 & 1 & 0 & 5\end{bmatrix} = \begin{bmatrix}5 & 1 & 6 & 9\\ -2 & -1 & 3 & -3\\ -3 & -2 & 9 & -4\end{bmatrix}.$$

#4.3 The three systems can be written as

$$\begin{bmatrix}2&-3&5\\-1&2&-3\\7&2&6\end{bmatrix}\begin{bmatrix}x_1&x_2&x_3\\y_1&y_2&y_3\\z_1&z_2&z_3\end{bmatrix}=\begin{bmatrix}-1&2&1\\0&-5&-3\\2&0&10\end{bmatrix}$$

or $AX=B$ with $A=\begin{bmatrix}2&-3&5\\-1&2&-3\\7&2&6\end{bmatrix}$, $X=\begin{bmatrix}x_1&x_2&x_3\\y_1&y_2&y_3\\z_1&z_2&z_3\end{bmatrix}$, $B=\begin{bmatrix}-1&2&1\\0&-5&-3\\2&0&10\end{bmatrix}$

#4.4. $B=\begin{bmatrix}-b_1-\\-b_2-\\-b_3-\end{bmatrix}$, $A=\begin{bmatrix}0&0&1\\0&-2&1\\-1&1&3\end{bmatrix}$

$0(1)+0(2)+1(3)\to(1)$
$0(1)-2(2)+1(3)\to(2)$
$-1(1)+1(2)+3(3)\to(3)$.

#4.5. $E\begin{bmatrix}1&3&0\\-1&0&5\\2&3&-3\end{bmatrix}=\begin{bmatrix}1&3&0\\0&3&5\\0&-3&-3\end{bmatrix}$

$(1)\to(1)$
$(2)+(1)\to(2)$
$(3)-2(1)\to(3)$

so $E=\begin{bmatrix}1&0&0\\1&1&0\\-2&0&1\end{bmatrix}$.

check: $E\begin{bmatrix}1&3&0\\-1&0&5\\2&3&-3\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&3&0\\-1&0&5\\2&3&-3\end{bmatrix}=\begin{bmatrix}1&3&0\\0&3&5\\0&-3&-3\end{bmatrix}$ ✓

#4.6. These problems were started in #3.2 (see HW#2).

(a) $\begin{bmatrix}2&4&1\\4&7&3\\2&5&6\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\1&-1&1\end{bmatrix}\begin{bmatrix}2&4&1\\0&-1&1\\0&0&6\end{bmatrix}$ (check this)

$$=\begin{bmatrix}1&0&0\\2&1&0\\1&-1&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&-1&0\\0&0&6\end{bmatrix}\begin{bmatrix}1&2&1/2\\0&1&-1\\0&0&1\end{bmatrix}=LDU.$$

(c) $\begin{bmatrix}1&2&3&4\\0&-1&2&-4\\2&3&-1&5\\-3&-1&3&-2\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\2&1&1&0\\-3&-5&-22/9&1\end{bmatrix}\begin{bmatrix}1&2&3&4\\0&-1&2&-4\\0&0&-9&1\\0&0&0&-68/9\end{bmatrix}=L\cancel{D}U.$

$$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\2&1&1&0\\-3&-5&-22/9&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-9&0\\0&0&0&-68/9\end{bmatrix}\begin{bmatrix}1&2&3&4\\0&1&-2&4\\0&0&1&-1/9\\0&0&0&1\end{bmatrix}=LDU.$$

#4.8(a) $2x_1 + 4x_2 + x_3 = 1$
$4x_1 + 7x_2 + 3x_3 = 1$
$2x_1 + 5x_2 + 6x_3 = -4$

or, in matrix form: $\begin{bmatrix} 2 & 4 & 1 \\ 4 & 7 & 3 \\ 2 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$.

Now, earlier we found:

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 7 & 3 \\ 2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LDU.$$

We solve $Ax = b$ in three (easy steps):

Step 1: Let $y = DUx$, then $b = Ax = L(DUx) = Ly$

so $y_1 = 1 \Rightarrow y_1 = 1$
$2y_1 + y_2 = 1 \Rightarrow y_2 = -1$
$y_1 - y_2 + y_3 = -4 \Rightarrow y_3 = -6$. so $y = \begin{bmatrix} 1 \\ -1 \\ -6 \end{bmatrix}$.

Step 2: Let $z = Ux$ then $y = DUx = Dz$

so $2y_1 = 1 \Rightarrow z_1 = 1/2$
$-y_2 = -1 \Rightarrow y_2 = 1$
$+6x_3 = -6$, $z_3 = -1$. so $z = \begin{bmatrix} 1/2 \\ 1 \\ -1 \end{bmatrix}$.

Step 3: Lastly, $z = Ux$

so $x_1 + 2x_2 + \frac{1}{2}x_3 = 1/2 \Rightarrow x_1 = 1$
$x_2 - x_3 = 1 \Rightarrow x_2 = 0$
$x_3 = -1 \Rightarrow x_3 = -1$

Thus, the solution is $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (& this checks!).