HW #2 – Solutions

Chapter 2

#2.1.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 7x_4 &= 0 \\ -2x_1 - x_2 + 4x_3 + 5x_4 &= 1 \\ 2x_1 + 3x_2 - 6x_3 + 4x_4 &= 3 \end{aligned} \iff \begin{bmatrix} 1 & 2 & -3 & 7 \\ -2 & -1 & 4 & 5 \\ 3 & 3 & -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

#2.2:

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & 3 & 5 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \iff \begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ -2x_1 + 3x_2 + 5x_3 &= -1 \\ 2x_1 + 3x_2 - 4x_3 &= 2 \end{aligned}$$

#2.3: There are many possibilities, e.g. Let A be a 2×1 matrix and B a 1×3 matrix. Then AB will be a 2×3 matrix and BA does not exist.

#2.4: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$

(a) $$(AB)C = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}.$$

(b) $$A(B+C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & -4 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 6 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & -4 \end{bmatrix}$$

(c) $$(B+C)A = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -9 & -10 \\ 5 & 6 \end{bmatrix}$$

$$BA + CA = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} -7 & -8 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -9 & -10 \\ 5 & 6 \end{bmatrix}$$

(d) $$(3A)B = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}, \quad A(3B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}$$

$$\& \; 3(AB) = 3 \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}$$

2.4 (cont.)

(e) $(AB)^t = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}^t = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$

$$A^tB^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \neq (AB)^t$$

$$B^tA^t = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = (AB)^t.$$

#2.5: $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$: $A^2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$$A^3 = A^2 A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$: $B^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$$B^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

#2.6: There are many solutions to this problem. If we think in terms of linear combinations, we might come up with an example like

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then $AB = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2\times2}.$

#2.7. To have $(AB)^2 \neq A^2B^2$, note that $(AB)^2 = ABAB$ and $A^2B^2 = AABB$. So, if $BA \neq AB$, then it's very likely that $(AB)^2 \neq A^2B^2$.

#2.7 $(AB)^2 = ABAB$ and $A^2B^2 = AABB$.

if $BA = AB$ then $A(BA)B = A(AB)B$, i.e. $(AB)^2 = A^2B^2$.

so, to have a chance that $(AB)^2 \neq A^2B^2$ we must have $BA \neq AB$.

The matrices A & B from #2.4 had $AB \neq BA$. Let's see if they work here.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \qquad B^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$(AB)^2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{so} \quad (AB)^2 \neq A^2B^2.$$

$$A^2B^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ -14 & 14 \end{bmatrix}$$

Note that if we used the matrices chosen in #2.6, we have $AB \neq BA$

but $(AB)^2 = A^2B^2$. This is because $AB = 0$ and $(AB)^2 = (AB)(AB) = 0^2 = 0$

$A^2B^2 = AABB = A(AB)B = A0B = 0.$

#2.8. This is answered in my comments to #2.7.

If $AB = BA$ then $(AB)^2 = A(BA)B = A(AB)B = A^2B^2$.

But, ~~also~~ as was also shown in #2.7, there are times when $AB \neq BA$

and we still have $(AB)^2 = A^2B^2$.

Chapter 3

#3.1(a) Backward substitution:

$$3x_1 - x_2 + 2x_3 = 3 \implies 3x_1 = 3,\ \underline{x_1 = 1}$$

$$5x_2 - x_3 = 9 \implies 5x_2 = 10,\ x_2 = 2$$

$$2x_3 = 2 \implies \underline{x_3 = 1}$$

so $x_1 = 1, x_2 = 2, x_3 = 1$.

(b) Forward substitution:

$$x_1 = 2 \implies x_1 = 2$$

$$2x_1 + 3x_2 = -1 \implies 3x_2 = -5,\ \underline{x_2 = -\tfrac{5}{3}}$$

$$-x_1 + 2x_2 - 4x_3 = 0 \implies -4x_3 = \tfrac{16}{3},\ \underline{\underline{x_3 = -\tfrac{4}{3}}}.$$

so $x_1 = 2, x_2 = -5/3, x_3 = -4/3$.

#3.2(a) $A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 7 & 3 \\ 2 & 5 & 6 \end{bmatrix}$: $\begin{bmatrix} & & \\ 2 & & \\ 1 & & \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 4 & 7 & 3 \\ 2 & 5 & 6 \end{bmatrix}$ $\quad (2) - 2(1) \to (2)$, $(3) - (1) \to (3)$

$$\begin{bmatrix} & & \\ 2 & & \\ 1 & & \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 5 \end{bmatrix} \quad (3) + (2) \to (3)$$

$$\begin{bmatrix} & & \\ 2 & & \\ 1 & -1 & \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

so $A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 7 & 3 \\ 2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 6 \end{bmatrix} = LU.$

(b) $A = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 4 & -1 \\ 3 & 3 & 4 \end{bmatrix}$: $\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ -2 & 4 & -1 \\ 3 & 3 & 4 \end{bmatrix}$ $\quad (2) + 2(1) \to (2)$, $(3) - 3(1) \to (3)$

$$\begin{bmatrix} & & \\ -2 & & \\ 3 & & \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 5 \\ 0 & 6 & -5 \end{bmatrix} \quad (3) - 3(2) \to (3)$$

$$\begin{bmatrix} & & \\ -2 & & \\ 3 & 3 & \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -20 \end{bmatrix}$$

#3.2 (b) (cont.)

$$\text{So } A = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 4 & -1 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -20 \end{bmatrix} = LU$$

(c)
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -4 \\ 2 & 3 & -1 & 5 \\ -3 & -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -4 \\ 2 & 3 & -1 & 5 \\ -3 & -1 & 3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ 0 & & & \\ 2 & & & \\ -3 & & & \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -4 \\ 0 & -1 & -7 & -3 \\ 0 & 5 & 12 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 2 & 1 & & \\ -3 & -5 & & \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -4 \\ 0 & 0 & -9 & 1 \\ 0 & 0 & 22 & -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 2 & 1 & & \\ -3 & -5 & -\frac{22}{9} & \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -4 \\ 0 & 0 & -9 & 1 \\ 0 & 0 & 0 & -68/9 \end{bmatrix}$$

$$\text{So } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -4 \\ 2 & 3 & -1 & 5 \\ -3 & -1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ -3 & -5 & -22/9 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -4 \\ 0 & 0 & -9 & 1 \\ 0 & 0 & 0 & -68/9 \end{bmatrix} = LU.$$

#3.3 (a) $Ax=b$ with
$$A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 7 & 3 \\ 2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 6 \end{bmatrix} = LU$$

$$Ax=b \text{ is } LUx = b = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

Let $y = Ux$ & solve $Ly = b$:
$$\begin{aligned} y_1 &= 1 &&\Rightarrow y_1 = 1 \\ 2y_1 + y_2 &= 1 &&\Rightarrow y_2 = -1 \\ y_1 - y_2 + y_3 &= -4 &&\Rightarrow y_3 = -6. \end{aligned}$$

$$\text{So } y = \begin{bmatrix} 1 \\ -1 \\ -6 \end{bmatrix}.$$

#3.3 (a) (cont.)

Then $Ux = y = \begin{bmatrix} 1 \\ -1 \\ -6 \end{bmatrix}$:

$$\begin{aligned} 2x_1 + 4x_2 + x_3 &= 1 \quad \Rightarrow 2x_1 = 2 \\ -x_2 + x_3 &= -1 \quad \Rightarrow -x_2 = 0 \\ 6x_3 &= -6 \Rightarrow x_3 = -1 \end{aligned}$$

so $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. (& this checks - do it!)

(b) $Ax = b$ with $b = \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 4 & -1 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -20 \end{bmatrix} = LU$.

Let $y = Ux$ so that $Ly = LUx = Ax = b$:

$Ly = b$:
$$\begin{aligned} y_1 &= 4 \Rightarrow y_1 = 4 \\ -2y_1 + y_2 &= 4 \Rightarrow y_2 = 12 \\ 3y_1 + 3y_2 + y_3 &= 8 \Rightarrow y_3 = -40. \end{aligned}$$
So $y = \begin{bmatrix} 4 \\ 12 \\ -40 \end{bmatrix}$.

Then $Ux = y$:
$$\begin{aligned} x_1 - x_2 + 3x_3 &= 4 \Rightarrow x_1 = -1 \\ 2x_2 + 5x_3 &= 12 \Rightarrow 2x_2 = 2,\ x_2 = 1 \\ -20x_3 &= -40 \Rightarrow x_3 = 2 \end{aligned}$$

so $x = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. (& this checks!)