MATH 526 (Section 1 & 2)
Prof. Meade

University of South Carolina
Fall 2005

Exam 2
October 25, 2005

Name: Key
SS # (last 4 digits): ______

Instructions:

1. There are a total of 4 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
5. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	56	
2	18	
3	16	
4	10	
Extra Credit	5+5	
Total	100	

Good Luck!

1. (56 points) [14 points each] For each of the following four matrices,

   (i) [4 points] Determine if the matrix ~~is singular or~~ nonsingular.

   (ii) [8 points] If the matrix is nonsingular matrix, find the inverse matrix.

   (iii) [2 points] Find κ_1 and κ_∞.

Also, A is lower triangular w/ no zero on the main diagonal so A is nonsingular.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

(i) A is nonsingular $\Leftrightarrow$ $ad-bc \neq 0$

Here, $ad-bc = 1(-1) - 1(0) = -1 \neq 0$

so A is nonsingular.

Check: $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

(ii) $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

(iii) $\kappa_1(A) = \|A\|_1 \|A^{-1}\|_1 = 2 \cdot 2 = 4$; $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 2 \cdot 2 = 4$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(i) $ad-bc = 4-4 = 0$ so B is singular.

(ii) B^{-1} does not exist.

(iii) $\kappa_1(B) = \cancel{\|B\|_1 \|B} \; \infty$
$\kappa_\infty(B) = \infty$
} because B is singular.

(c) $\mathbf{C} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & 1 \\ 2 & 5 & 1 \end{bmatrix}$

(i) Find the LU factorization:

$$\begin{bmatrix} & & \\ -1 & & \\ 2 & & \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} & & \\ -1 & & \\ 2 & -1/2 & \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{bmatrix}$

& $U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

because U has a zero on the diagonal,

C is singular.

Check: $LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & 1 \\ 2 & 5 & 1 \end{bmatrix} = C$ ✓

(ii) C^{-1} does not exist

(iii) $\kappa_1(C) = \kappa_\infty(C) = \infty$

(i) [4 points] Determine if the matrix is singular or nonsingular.

(ii) [8 points] If the matrix is nonsingular matrix, find the inverse matrix.

(iii) [2 points] Find κ_1 and κ_∞.

(d) $\mathbf{E} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (i) Find the LU factorization of E: $\begin{bmatrix} 1 & & \\ 0 & & \\ 1 & & \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

so $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

& $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

because U does not have a 0 on the diagonal, E is non-singular.

(ii) To find E^{-1}: 1) $Ex = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$: $Ly = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$Ux = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

2) $Ex = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$: $Ly = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$Ux = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$

3) $Ex = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$: $Ly = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$Ux = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

so $E^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

Check: $EE^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3\times3}$ ✓

(iii) $\|E\|_1 = 4$ $\quad \|E^{-1}\|_1 = 5$ $\quad$ so $\kappa_1(E) = 20$

$\|E\|_\infty = 5$ $\quad \|E^{-1}\|_\infty = 4$ $\quad \kappa_2(E) = 20.$

2. (18 points)

(a) [14 points] Use the ~~tridiagonal~~ algorithm to find the L–U factorization of

$$\mathbf{F} = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 2 & 6 & 4 & 0 \\ 0 & 5 & 15 & -10 \\ 0 & 0 & -11 & -33 \end{bmatrix}.$$

$$\begin{bmatrix} -2 & -1 & 0 & 0 \\ -1\ \cancel{2} & \overset{5}{\cancel{6}} & 4 & 0 \\ 0 & {}^{1}\cancel{5} & \overset{11}{\cancel{15}} & -10 \\ 0 & 0 & \underset{-1}{-\cancel{11}} & \underset{-43}{-\cancel{33}} \end{bmatrix}$$

so $F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & 11 & -10 \\ 0 & 0 & 0 & -43 \end{bmatrix}$

$$= \begin{bmatrix} -2 & -1 & 0 & 0 \\ 2 & 6 & 4 & 0 \\ 0 & 5 & 15 & -10 \\ 0 & 0 & -11 & -33 \end{bmatrix}$$

(b) [4 points] Is **F** column diagonally dominant? (Explain.)

column 1: $|-2| \geq |2| + |0| + |0|$

2: $|6| \geq |-1| + |5| + |0|$

3: $|15| \geq |0| + |4| + |-11|$

4: $|-33| \geq |0| + |0| + |-10|$

so, Yes F is column diagonally dominant.

Extra Credit [5 points] What other conditions, other than column diagonally dominant, are needed to assure that Gaussian elimination applied to a tridiagonal matrix can be completed without pivoting? HINT: *There are two conditions.*

1) the entries in the superdiagonal must all be non-zero.

2) $|a_{21}| > |a_{11}|$

3. (16 points) [4 points each] Let $\mathbf{A} = \begin{bmatrix} 1.0 & 0.9 & -0.1 \\ 0.5 & 1.0 & 0.5 \\ 0.1 & 0.9 & 1.0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{x}' = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(a) Find the residual.

$$b' = Ax' = \begin{bmatrix} 2+0-0.1 \\ 1+0+0.5 \\ 0.2+0+1 \end{bmatrix} = \begin{bmatrix} 1.9 \\ 1.5 \\ 1.2 \end{bmatrix}$$

$$r = b - b' = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1.9 \\ 1.5 \\ 1.2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.5 \\ 0.8 \end{bmatrix}.$$

(b) Find the residual equation. *Do not solve!*

$$Ae = r$$

(c) Compute the relative error of the right-hand side in the ∞-norm.

$$\frac{\|b-b'\|_\infty}{\|b\|_\infty} = \frac{\|r\|_\infty}{\|b\|_\infty} = \frac{0.8}{2} = 0.4$$

(d) Explain why the relative error in $\mathbf{x}'$, in the ∞-norm, cannot exceed 15.2.

NOTES: *Use $\kappa_\infty(A) = 38$. Do <u>not</u> even attempt to find $\mathbf{A}^{-1}$ or $\mathbf{x}$.*

$$\frac{\|x-x'\|_\infty}{\|x\|_\infty} \leq \kappa_\infty(A) \frac{\|b-b'\|_\infty}{\|b\|_\infty} = 38(0.4) = 15.2$$

Extra Credit [5 points] Without finding $\mathbf{A}^{-1}$, find $\|\mathbf{A}^{-1}\|_\infty$.

$$\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty \quad \text{so} \quad \|A^{-1}\|_\infty = \frac{\kappa_\infty(A)}{\|A\|_\infty} = \frac{38}{2} = 19$$

4. (10 points) [5 points each] Prove any <u>two</u> of the following three statements:

   (a) **Claim:** *For any $n \times n$ matrix* $\mathbf{A}$, $\|A\|_\infty = \|A^t\|_1$.

   (b) **Claim:** *For any $n \times n$ matrix* $\mathbf{A}$, $\frac{1}{n}\|A\|_\infty \le \|A\|_1 \le n\|A\|_\infty$.

   HINT: Use the fact that $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_1 \le n\|\mathbf{x}\|_\infty$.

   (c) **Claim:** *For any matrix norm* $\|\cdot\|$ *and any $n \times n$ matrices* $\mathbf{A}$ *and* $\mathbf{B}$, $\|\mathbf{AB}\| \le \|\mathbf{A}\|\|\mathbf{B}\|$.

   HINT: Use the fact that $\|\mathbf{Ax}\| \le \|\mathbf{A}\|\|\mathbf{x}\|$.

Be sure to clearly indicate which statements you are proving.

HINT: *Start each proof by restating the claim.*

Claim: For any $n\times n$ matrix A, $\|A\|_\infty = \|A^t\|_1$.

Proof: Let A be an $n \times n$ matrix. Because the rows of A are the columns of A^t,

$$\|A^t\|_1 = \text{max absolute column sum of } A^t = \text{max absolute row sum of } A = \|A\|_\infty. \ \blacksquare$$

Claim: For any $n\times n$ matrix A, $\frac{1}{n}\|A\|_\infty \le \|A\|_1 \le n\|A\|_\infty$.

Proof: Let A be an $n\times n$ matrix. We know that for any vector u, $\|u\|_\infty \le \|u\|_1 \le n\|u\|_\infty$.

$$\|A\|_1 = \max_{x\ne 0} \frac{\|Ax\|_1}{\|x\|_1} \ \& \ \|A\|_\infty = \max_{x \ne 0}\frac{\|Ax\|_\infty}{\|x\|_\infty}.$$

$$\frac{1}{n}\frac{\|Ax\|_\infty}{\|x\|_\infty} = \frac{\|Ax\|_\infty}{n\|x\|_\infty} \le \frac{\|Ax\|_1}{\|x\|_1} \le \frac{n\|Ax\|_\infty}{\|x\|_\infty} = n\frac{\|Ax\|_\infty}{\|x\|_\infty} \quad \text{for any vector } x \ne 0.$$

Taking the max over all nonzero vectors x:

$$\frac{1}{n}\max \frac{\|Ax\|_\infty}{\|x\|_\infty} \le \max_{x\ne0}\frac{\|Ax\|_1}{\|x\|_1} \le n \max_{x \ne 0}\frac{\|Ax\|_\infty}{\|x\|_\infty}$$

so $\frac{1}{n}\|A\|_\infty \le \|A\|_1 \le n\|A\|_1$. $\blacksquare$

Claim: For any matrix norm $\|\cdot\|$ & any $n\times n$ matrices A & B, $\|AB\| \le \|A\|\|B\|$.

Proof: Let $\|\ \|$ be a matrix norm and A & B be matrizes $(n\times n)$.

$$\|AB\| = \max_{x\ne0}\frac{\|ABx\|}{\|x\|} = \max_{x\ne0}\frac{\|A(Bx)\|}{\|x\|} \le \max_{x\ne0}\frac{\|A\|\|Bx\|}{\|x\|} \le \max_{x\ne0}\frac{\|A\|\|B\|\|x\|}{\|x\|} = \|A\|\|B\|\max_{x\ne0} 1 = \|A\|\|B\|. \ \blacksquare$$

because $\|Au\| \le \|A\|\|u\|$ (with $u = Bx$)

because $\|Bx\| \le \|B\|\|x\|$.

Extra Credit (5 points) Prove the remaining statement in Problem 4.

Put your proof on the back of this page.