MATH 526 (Section 1 & 2)
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University of South Carolina
Fall 2005

Exam 1
September 21, 2005

Name: Key
SS # (last 4 digits): __________

Instructions:

1. There are a total of 6 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
5. Check your work. If I see *clear evidence* that you checked your answer (when possible) <u>and</u> you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	24	
2	25	
3	16	
4	15	
5	10	
6	10	
Total	100	

Good Luck!

1. (24 points) [4 points each] Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

Compute each of the following quantities. *If an expression does not exist, indicate why.*

(a) **ABC** $BC = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$ABC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 8 \\ 4 & 11 & 18 \end{bmatrix}$

(b) **ACB** CB is not defined; C has 3 columns & B has 2 rows.
so ACB is not defined.

(c) $\mathbf{C}^t\mathbf{B}^t = (BC)^t$

From (a), $BC = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

so $C^tB^t = (BC)^t = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$.

(d) $(\mathbf{AC}^t)^t$ A is 2×2 and C^t is 3×2 so AC^t is not defined.
& so $(AC^t)^t$ is not defined.

(e) $\mathbf{v}^t\mathbf{Cw}$ $Cw = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ -12 \end{bmatrix}$

$v^tCw = \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} 16 \\ -12 \end{bmatrix} = \begin{bmatrix} -12 \end{bmatrix}$

(f) $\mathbf{wv}^t$

w is 3×1
v^t is 1×2, so wv^t will be a 3×2 matrix.

$wv^t = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 12 & 20 \\ 18 & 30 \end{bmatrix}$

2. (25 points)

(a) [15 points] Express the following linear combinations of the row vectors $\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 & 2 \end{bmatrix}$, and $\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$ in terms of matrix multiplication.

$$2\begin{bmatrix} 1 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 6 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 & 2 \end{bmatrix} + 2\begin{bmatrix} -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 6 \\ -1 & 0 & 2 \end{bmatrix}.$$

(b) [10 points] Find the 3×3 matrix $\mathbf{E}$ such that

$$\mathbf{E}\begin{bmatrix} 3 & 7 & -2 \\ -3 & -5 & 1 \\ 6 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 7 & -2 \\ 0 & 2 & -1 \\ 0 & -9 & 4 \end{bmatrix}.$$

HINT: *Think about row operations.*

$$\begin{bmatrix} 3 & 7 & -2 \\ -3 & -5 & 1 \\ 6 & 5 & 0 \end{bmatrix} \xrightarrow[-2(1)+(3) \to (3)]{(1) \to (1) \quad (1)+(2) \to (2)} \begin{bmatrix} 3 & 7 & -2 \\ 0 & 2 & -1 \\ 0 & -9 & 4 \end{bmatrix}$$

so $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$

3. (16 points) The matrices in the **L**–**U** factorization of a matrix **A** are

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find the solution to $\mathbf{Ax} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$.

We know $A = LU$, so $Ax = b$ when $LUx = b$.
To solve this, let $Ux = y$, then $Ly = L(Ux) = Ax = b$.

① forward : $Ly = b$

$$\begin{aligned} y_1 &= 1 \Rightarrow y_1 = 1 \\ -3y_1 + y_2 &= -2 \Longrightarrow y_2 = -2 + 3y_1 = -2+3 = 1 \\ 3y_1 - 2y_2 + y_3 &= -1 \Longrightarrow y_3 = -1 - 3y_1 + 2y_2 = -1-3+2 = -2 \\ -5y_1 + 4y_2 - y_3 + y_4 &= 2 \Longrightarrow y_4 = 2 + 5y_1 - 4y_2 + y_3 = 2+5-4-\cancel{2} = 1 \end{aligned}$$

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so $y = \begin{bmatrix} 1 \\ 1 \\ -2 \\ +\cancel{3}1 \end{bmatrix}$ Check : $Ly = \begin{bmatrix} 1 & & & \\ -3 & 1 & & \\ 3 & -2 & 1 & \\ -5 & 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ +\cancel{3}1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ +2 \end{bmatrix}$ ✓

② Backward: $Ux = y$

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &= 1 \Longrightarrow x_1 = 1 - 3x_2 - 4x_3 = 1+6-4 = 3 \\ 3x_2 + 5x_3 + 2x_4 &= 1 \Longrightarrow 3x_2 = 1 - 5x_3 - 2x_4 = -6,\ x_2 = -2 \\ -2x_3 &= -2 \Longrightarrow x_3 = 1 \\ x_4 &= 1 \Rightarrow x_4 = 1 \end{aligned}$$

so $x = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}$. Check: $Ux = \begin{bmatrix} 1 & 3 & 4 & 0 \\ & 3 & 5 & 2 \\ & & -2 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} = y$ ✓

4. (15 points) Let $\mathbf{A} = \begin{bmatrix} -1 & 1 & -4 \\ 2 & 2 & -6 \\ 3 & 0 & -3 \end{bmatrix}$. Perform Gaussian elimination with partial pivoting on $\mathbf{A}$. What is the $\mathbf{L}$–$\mathbf{U}$–$\mathbf{P}$ factorization of $\mathbf{A}$? *Be sure you clearly identify* $\mathbf{L}$, $\mathbf{U}$, *and* $\mathbf{P}$.

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} -1 & 1 & -4 \\ 2 & 2 & -6 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) → (1)
(1) → (3)

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & -6 \\ -1 & 1 & -4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(2) $- \frac{2}{3}$(1) → (2)
(3) $- (-\frac{1}{3})$(1) → (3)

$$\begin{bmatrix} & & \\ 2/3 & & \\ -1/3 & & \end{bmatrix} \begin{bmatrix} 3 & 0 & -3 \\ 0 & 2 & -4 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(3) $- \frac{1}{2}$(2) → (3)

$$\begin{bmatrix} & & \\ 2/3 & & \\ -1/3 & 1/2 & \end{bmatrix} \begin{bmatrix} 3 & 0 & -3 \\ 0 & 2 & -4 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

so $L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ -1/3 & 1/2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & 0 & -3 \\ 0 & 2 & -4 \\ 0 & 0 & -3 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Check: $LU = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ -1/3 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -3 \\ 0 & 2 & -4 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & -6 \\ -1 & 1 & -4 \end{bmatrix}$ ✓

$PA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & -4 \\ 2 & 2 & -6 \\ 3 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & -6 \\ -1 & 1 & -4 \end{bmatrix}$

5. (10 points) Find two 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \neq \mathbf{0}$, $\mathbf{B} \neq \mathbf{0}$, and $\mathbf{AB} = \mathbf{0}$.

The most common answers are: $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

but any two matrizes that are nonzero & whose product is 0 is accepted.

6. (10 points) Count the floating-point operations (multiplication and addition) needed to compute the product of an upper triangular $n \times n$ matrix and a column n-vector.

Do not count multiplication by zero or addition of zero.

$$\begin{bmatrix} * & * & * & \cdots & * \\ & \ddots & & & \vdots \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{bmatrix} \begin{bmatrix} * \\ * \\ * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} ** + ** + ** + \cdots + ** \\ ** + ** + \cdots + ** \\ \vdots \\ ** + ** \\ ** \end{bmatrix} \qquad \begin{array}{cc} \#\text{ mult} & \#\text{ add} \\ n & n-1 \\ n-1 & n-2 \\ \vdots & \vdots \\ 2 & 1 \\ 1 & 0 \end{array}$$

$$\sum_{k=1}^{n} k + \sum_{k=0}^{n-1} k$$

$$= \frac{n(n+1)}{2} + \frac{(n-1)n}{2}$$

$$= \frac{n}{2}(n+1+n-1)$$

$$= \frac{n}{2}(2n)$$

$$= \underline{\underline{n^2 \text{ flops}}}$$