>

Maple Solution to the Extra Credit Problem -- Math 524

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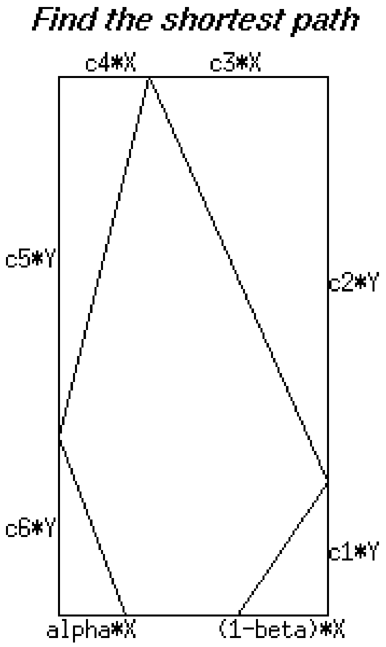
The original problem, posed by Andrey Andreev to Kostya Oskolkov, is to find the shortest path between two points on the same side of a rectangle with the constraint that the path must touch all four sides of the rectangle. The following figure gives the general picture, and defines the notation used throughout the remainder of this worksheet.

```
> with( plots ):
  setoptions( axes=NONE, scaling=CONSTRAINED );
> p1 := plot( { [ [0,0], [1,0], [1,2], [0,2], [0,0] ], [ [2/3, 0], [1,1/2], [1/3,2], [0,2/3], [1/4,0] ] } ):
  p2 := textplot( { [1/8,0,`alpha*X`], [5/6,0,`(1-beta)*X`] }, align={BELOW,CENTER} ):
  p3 := textplot( { [1,1/4,`c1*Y`], [1,5/4,`c2*Y`] }, align={RIGHT,CENTER} ):
  p4 := textplot( { [2/3,2,`c3*X`], [1/5,2,`c4*X`] }, align={ABOVE,CENTER} ):
  p5 := textplot( { [0,4/3,`c5*Y`], [0,1/3,`c6*Y`] }, align={LEFT,CENTER} ):
  display( { 'p.k' $ 'k'=1..5 }, title=`Find the shortest path` );
```



>

```
> restart;
  with( linalg ):
Warning: new definition for   norm
Warning: new definition for   trace
```

The mathematical formulation of this problem is a constrained optimization problem. Given values for the parameters X, Y > 0 and 0 <= alpha <= beta <= 1 the total distance travelled

```
> f:=sqrt( ((1-beta)*X)^2 + (c1*Y)^2 )
   + sqrt( (c3*X)^2 + (c2*Y)^2 )
   + sqrt( (c4*X)^2 +(c5*Y)^2 )
   + sqrt( (alpha*X)^2+(c6*Y)^2 );
```

$$f := \sqrt{X^2 - 2\,X^2\,\beta + X^2\,\beta^2 + c1^2\,Y^2} + \sqrt{c3^2\,X^2 + c2^2\,Y^2} + \sqrt{c4^2\,X^2 + c5^2\,Y^2} + \sqrt{\alpha^2\,X^2 + c6^2\,Y^2}$$

as a function of the six variables

```
> allvars := c1, c2, c3, c4, c5, c6:
```

There are three constraints:
c1 + c2 = 1
c3 + c4 = 1
c5 + c6 = 1

Note that the assumption that alpha <= beta is made so that the moving towards the right is the correct first step. If beta < alpha, simply reverse the direction of the path. A full justification of this assumption will be evident from the geometric reasoning at the end of this worksheet.

Instead of treating this as a problem of six variables, it seems more reasonable to keep only c1, c3, and c5. Thus,

```
> redvars := c1, c3, c5:
> reduce := c2=1-c1, c4=1-c3, c6=1-c5:
> f := subs( reduce, f );
```

$$f := \sqrt{X^2 - 2\,X^2\,\beta + X^2\,\beta^2 + c1^2\,Y^2} + \sqrt{c3^2\,X^2 + (1 - c1)^2\,Y^2} + \sqrt{(1 - c3)^2\,X^2 + c5^2\,Y^2} + \sqrt{\alpha^2\,X^2 + (1 - c5)^2\,Y^2}$$

```
>
```

We now seek the critical points of f and will then attempt to classify one as a global minimizer. For this we need the gradient and Hessian of f (as a function of c1, c3, c5).

```
> df := grad( f, [redvars] );
```

$$df := \left[\frac{c1\,Y^2}{\sqrt{X^2 - 2\,X^2\,\beta + X^2\,\beta^2 + c1^2\,Y^2}} - \frac{(1 - c1)\,Y^2}{\sqrt{c3^2\,X^2 + (1 - c1)^2\,Y^2}} \quad \frac{c3\,X^2}{\sqrt{c3^2\,X^2 + (1 - c1)^2\,Y^2}} - \frac{(1 - c3)\,X^2}{\sqrt{(1 - c3)^2\,X^2 + c5^2\,Y^2}} \quad \frac{c5\,Y^2}{\sqrt{(1 - c3)^2\,X^2 + c5^2\,Y^2}} - \frac{(1 - c5)\,Y^2}{\sqrt{\alpha^2\,X^2 + (1 - c5)^2\,Y^2}} \right]$$

```
> Hf := hessian( f, [redvars] );  # you don't really want to see this!
>
```

Find all critical points

```
> cp := [ solve( { df[1]=0, df[2]=0, df[3]=0 }, { redvars } ) ];
```

$$cp := \left[\left\{ c5 = -\frac{\beta+\alpha}{-\beta+\alpha}, c1 = -2\,\frac{\beta-1}{-\beta+\alpha}, c3 = -\frac{1}{2}\alpha + 1 - \frac{1}{2}\beta \right\}, \left\{ c5 = -\frac{-\beta+\alpha}{\beta+\alpha}, c3 = \frac{1}{2}\alpha - \frac{1}{2}\beta + 1, c1 = 2\,\frac{\beta-1}{\beta+\alpha} \right\}, \left\{ c1 = -2\,\frac{\beta-1}{\alpha-\beta+2}, c3 = \frac{1}{2}\beta + \frac{1}{2}\alpha, c5 = -\frac{\alpha-2+\beta}{\alpha-\beta+2} \right\}, \left\{ c3 = \frac{1}{2}\beta - \frac{1}{2}\alpha, c5 = -\frac{\alpha-\beta+2}{\alpha-2+\beta}, c1 = 2\,\frac{\beta-1}{\alpha-2+\beta} \right\} \right]$$

Look closely at these critical points. Which, if any, are feasible (i.e. satisfy the constraints). To assist with this, let's look at the full set of parameter values (c1, c2, c3, c4, c5, c6) for each of the critical points.

```
> map( normal, subs( reduce, cp[1], [ allvars ] ) );
```

$$\left[-2\,\frac{\beta-1}{-\beta+\alpha}, \frac{\alpha-2+\beta}{-\beta+\alpha}, -\frac{1}{2}\alpha + 1 - \frac{1}{2}\beta, \frac{1}{2}\beta + \frac{1}{2}\alpha, -\frac{\beta+\alpha}{-\beta+\alpha}, 2\,\frac{\alpha}{-\beta+\alpha} \right]$$

```
> map( normal, subs( reduce, cp[2], [ allvars ] ) );
```

$$\left[2\,\frac{\beta-1}{\beta+\alpha}, \frac{\alpha-\beta+2}{\beta+\alpha}, \frac{1}{2}\alpha - \frac{1}{2}\beta + 1, \frac{1}{2}\beta - \frac{1}{2}\alpha, -\frac{-\beta+\alpha}{\beta+\alpha}, 2\,\frac{\alpha}{\beta+\alpha} \right]$$

```
> map( normal, subs( reduce, cp[3], [ allvars ] ) );
```

$$\left[-2\,\frac{\beta-1}{\alpha-\beta+2}, \frac{\beta+\alpha}{\alpha-\beta+2}, \frac{1}{2}\beta + \frac{1}{2}\alpha, -\frac{1}{2}\alpha + 1 - \frac{1}{2}\beta, -\frac{\alpha-2+\beta}{\alpha-\beta+2}, 2\,\frac{\alpha}{\alpha-\beta+2} \right]$$

```
> map( normal, subs( reduce, cp[4], [ allvars ] ) );
```

$$\left[2\,\frac{\beta-1}{\alpha-2+\beta}, \frac{-\beta+\alpha}{\alpha-2+\beta}, \frac{1}{2}\beta - \frac{1}{2}\alpha, \frac{1}{2}\alpha - \frac{1}{2}\beta + 1, -\frac{\alpha-\beta+2}{\alpha-2+\beta}, 2\,\frac{\alpha}{\alpha-2+\beta} \right]$$

Some useful, and easy, facts are:
2 - alpha - beta > 0   2 + alpha - beta > 0   2 - alpha + beta > 0

For one critical point c1<0 and for two critical points c6<0 (for any valid choice of alpha and beta). The remaining critical point is feasible; all six variables are positive for all values of alpha and beta that satisfy the original assumption.

The exact order of these points may change from one run to the next (the elements of a set are unordered). Identify which of the above critical points is feasible and correct, if necessary, the index on the next line:

```
> CP := cp[ 3 ];
```

$$CP := \left\{ c1 = -2\,\frac{\beta-1}{\alpha-\beta+2}, c3 = \frac{1}{2}\beta + \frac{1}{2}\alpha, c5 = -\frac{\alpha-2+\beta}{\alpha-\beta+2} \right\}$$

```
>
```

But, is this really the minimizer?

To answer this we might consider examining the Hessian

```
> map( x->simplify(subs( CP, x ), symbolic ), Hf );
```

$$\left[\frac{(\alpha-\beta+2)\,Y^2X^2\left(\alpha^2\beta-5\,\alpha\,\beta^2+12\,\alpha\,\beta+3\,\beta^3-14\,\beta^2+20\,\beta+\alpha^3+2\,\alpha^2-4\,\alpha-8\right)}{(\beta-1)\,\%1^{3/2}\,(\beta+\alpha)},\ 4\,\frac{Y^2X^2(\alpha-\beta+2)^2}{\%1^{3/2}\,(\beta+\alpha)},\ 0\right]$$

$$\left[4\,\frac{Y^2X^2(\alpha-\beta+2)^2}{\%1^{3/2}\,(\beta+\alpha)},\ 16\,\frac{(\alpha-\beta+2)\,X^2Y^2(\alpha-1+\beta)}{(\beta+\alpha)\,\%1^{3/2}\,(\alpha-2+\beta)},\ 4\,\frac{X^2Y^2(\alpha-\beta+2)^2}{(\alpha-2+\beta)\,\%1^{3/2}}\right]$$

$$\left[0,\ 4\,\frac{X^2Y^2(\alpha-\beta+2)^2}{(\alpha-2+\beta)\,\%1^{3/2}},\ \frac{(\alpha-\beta+2)\,Y^2X^2\left(3\,\alpha^3-5\,\alpha^2\beta+10\,\alpha^2+\alpha\,\beta^2-4\,\alpha\,\beta+4\,\alpha-6\,\beta^2+12\,\beta-8+\beta^3\right)}{(\alpha-2+\beta)\,\%1^{3/2}\,\alpha}\right]$$

$$\%1 := \alpha^2X^2-2\,X^2\alpha\,\beta+4\,X^2\alpha+X^2\beta^2-4\,X^2\beta+4\,X^2+4\,Y^2$$

It is not reasonable to expect this to yield anything useful. (The eigenvalues will be impossible to compute and the prinicipal minors are no simpler!) Convexity doesn't seem to apply (the square root function is concave) and coercivity is out of the question because this is not an unconstrained problem. All we can say is that the minimizer is either this critical point, or a point on the boundary of the feasible set. First, the total distance when the nodes are chosen to be the feasible critical point:

```
> simplify( subs( CP, f ) );
```

$$\sqrt{\frac{(\beta-1)^2\,\%1}{(\alpha-\beta+2)^2}}+\frac{1}{2}\sqrt{\frac{(\beta+\alpha)^2\,\%1}{(\alpha-\beta+2)^2}}+\frac{1}{2}\sqrt{\frac{(\alpha-2+\beta)^2\,\%1}{(\alpha-\beta+2)^2}}+\sqrt{\frac{\alpha^2\,\%1}{(\alpha-\beta+2)^2}}$$

$$\%1 := \alpha^2X^2-2\,X^2\alpha\,\beta+4\,X^2\alpha+X^2\beta^2-4\,X^2\beta+4\,X^2+4\,Y^2$$

Note that the common expression (%1) can be written as
*(2 - (beta-alpha))^2 * X^2 + 4 * Y^2*
The relevance of this will become apparent when the geometry of this solution is examined.

```
>
```

The boundary of the feasible region consists of the six sides of the unit cube:
c1 = 0, 0 <= c3, c5 <= 1
c1 = 1, 0 <= c3, c5 <= 1
c3 = 0, 0 <= c1, c5 <= 1
c3 = 1, 0 <= c1, c5 <= 1
c5 = 0, 0 <= c1, c3 <= 1
c5 = 1, 0 <= c1, c3 <= 1
Checking all of these sides is a rather long and tedious exercise, as if this has not been long and tedious already!).

To conclude, we should notice some nice geometry in this solution. Each segment of the path creates a right triangle. The four triangles are all similar; in particular, their side lengths of one triangle are a multiple of the side lengths of the other triangles. Another way of stating this, which is less geometric but more physical, is that the incoming and outgoing angles of reflection are equal.

Along the right edge the angles satisfy

```
> tan( theta[right] ) = simplify( subs( reduce, CP, (1-beta)*X / (c1*Y) ) );
```

$$\tan(\theta_{right}) = \frac{1}{2}\frac{(\alpha - \beta + 2)X}{Y}$$

```
> tan( theta[right] ) = simplify( subs( reduce, CP, c3*X / (c2*Y) ) );
```

$$\tan(\theta_{right}) = \frac{1}{2}\frac{(\alpha - \beta + 2)X}{Y}$$

Along the top edge the angles satisfy

```
> tan( theta[top] ) = simplify( subs( reduce, CP, c2*Y / (c3*X) ) );
```

$$\tan(\theta_{top}) = 2\frac{Y}{(\alpha - \beta + 2)X}$$

```
> tan( theta[top] ) = simplify( subs( reduce, CP, c5*Y / (c4*X) ) );
```

Along the left edge the angles satisfy

```
> tan( theta[left] ) = simplify( subs( reduce, CP, c4*X / (c5*Y) ) );
```

$$\tan(\theta_{left}) = \frac{1}{2}\frac{(\alpha - \beta + 2)X}{Y}$$

```
> tan( theta[left] ) = simplify( subs( reduce, CP, alpha*X / (c6*Y) ) );
```

Even the angles along the bottom edge are equal

```
> tan( theta[finish] ) = simplify( subs( reduce, CP, c6*Y / (alpha*X) ) );
```

$$\tan(\theta_{finish}) = 2\frac{Y}{(\alpha - \beta + 2)X}$$

```
> tan( theta[start] ) = simplify( subs( reduce, CP, c1*Y / ( (1-beta)*X) ) );
```

$$\tan(\theta_{start}) = 2\frac{Y}{(\alpha - \beta + 2)X}$$

```
>
```

It should also be noted that the angles on the left and right edges are equal, as are the initial, top, and terminal angles. Moreover, note the reciprocal relationship between the angles along the vertical and horizontal edges. These geometric facts can be very useful if you wish to consider extensions of this problem, e.g., the shortest path that touches all three sides of an arbitrary triangle, pentagon, etc.

Also, the geometric properties of the solution suggest that it would also be reasonable to formulate this problem in terms of the angles made with the sides of the rectangle. This requires a little more trigonometry, but nothing too fancy.

To close, it is a relatively simple matter to see the solution for any values of the parameters X, Y, alpha, and beta. Simply set the values in the next command, then create the plot. Change the parameter values, and note the geometric properties that are present in all solutions.

```
> PARAM := X = 1, Y = 2, alpha = 1/2, beta = 2/3:
```

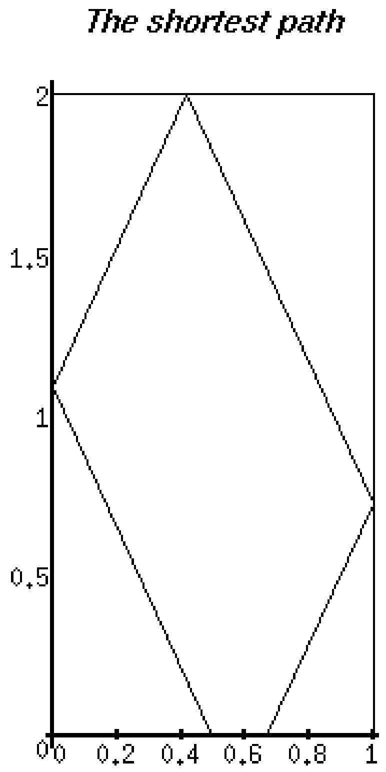
The values for the six variables c1, c2, c3, c4, c5, c6 are

```
> subs( reduce, CP, PARAM, [ allvars ] );
```

$$\left[\frac{4}{11}, \frac{7}{11}, \frac{7}{12}, \frac{5}{12}, \frac{5}{11}, \frac{6}{11}\right]$$

The sketch of this solution would be

```
> plot( subs( CP, PARAM,
      { [ [0,0], [X,0], [X,Y], [0,Y], [0,0] ],
        [ [beta*X, 0], [X,c1*Y], [(1-c3)*X,Y], [0,(1-c5)*Y], [alpha*X,0] ] } ),
      title=‘The shortest path‘, scaling=CONSTRAINED );
```



I encourage you to look at solutions for different values of the parameters. Particular questions you might wish to ponder include: how many solutions exist when alpha = beta? why isn’t this solution optimal when alpha > beta?

ENJOY!

END OF WORKSHEET

```
>
```