> restart:
with(plots):

Maple Worksheet to Accompany Problem #8ab, Chapter 2 (p. 81)
of Peressini, Sullivan, and Uhl

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a.
There are several ways to prove that the strict global minimizer of f is x=1 (2nd derivative test, convexity, coercivity, A-GMI). Before moving on to the next problems, let's take a moment ot look at the graph of the function.

> f := x+1/x;

$$f := x + \frac{1}{x}$$

> plot(f, x=0..10);
>

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b.
*Note that the function g can be written as: g(x1,x2) = f(2*x1^2 + x2^2). Now, using our knowledge from a. :*
*g(x1, x2) = f(2*x1^2 + x2^2) >= f(1) = 2*
*That is, the minimum value of g is 2 and this value is attained whenever 2*x1^2+x2^2=1.*

Let's verify these results by looking at a graph of g
> g := (2*x1^2+x2^2) + 1/(2*x1^2+x2^2);

$$g := 2\,x1^2 + x2^2 + \frac{1}{2\,x1^2 + x2^2}$$

> plot3d(g, x1=-2..2, x2=-2..2, view=1..10, labels=['x1','x2','z'], axes=BOXED, style=PATCHCONTOUR, shading=ZHUE);
Rotate the plot until you can view the graph from the top (or bottom). Why is the hole in the center black? Note the shape of the contours. Use the color information to identify the MINIMIZERS of g.

Now, rotate the graph until you are looking at only the x1 and z axes. What is the minimum VALUE of g?

If you want slightly different views of the function, try playing with the domain or range used in the plot. Here are two examples to get you started:
> plot3d(g, x1=-1..1, x2=-1..1, view=1..10, labels=['x1','x2','z'], axes=BOXED, style=PATCHCONTOUR, shading=ZHUE);
> plot3d(g, x1=-2..2, x2=-2..2, view=1.5..3, labels=['x1','x2','z'], axes=BOXED, style=PATCHCONTOUR, shading=ZHUE);
>

Think about what would need to be done to solve this problem using gradients and Hessians.

> with(linalg):

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Warning: new definition for   norm
Warning: new definition for   trace
```

> G := grad(g, [x1,x2]);

$$G := \left[4\,x1 - 4\,\frac{x1}{(2\,x1^2 + x2^2)^2} \quad 2\,x2 - 2\,\frac{x2}{(2\,x1^2 + x2^2)^2} \right]$$

> solve({ G[1]=0, G[2]=0 }, {x1,x2});

$$\{x2 = 1, x1 = 0\}, \{x1 = 0, x2 = -1\}, \{x1 = 0, x2 = \mathrm{RootOf}(_Z^2 + 1)\}, \left\{x1 = x1, x2 = \sqrt{1 - 2\,x1^2}\right\},$$
$$\left\{x1 = x1, x2 = -\sqrt{1 - 2\,x1^2}\right\}, \left\{x1 = x1, x2 = \sqrt{-1 - 2\,x1^2}\right\}, \left\{x1 = x1, x2 = -\sqrt{-1 - 2\,x1^2}\right\},$$
$$\{x1 = \mathrm{RootOf}(-1 + 2\,_Z^2), x2 = 0\}, \{x1 = \mathrm{RootOf}(1 + 2\,_Z^2), x2 = 0\}$$

>

*Careful inspection of these results leads to the conclusion that the critical points are the points (x1,x2) that satisfy 2*x1^2+x2^2 = 1. But, are these really minimizers? For this we need to look at the Hessian of g (alternatively, you could determine that g is a convex function. But, supposing you continued with the brute force approach, the Hessian matrix is:*

> H := hessian(g, [x1,x2]);

$$H := \left[\begin{array}{cc} 4 + 32\,\frac{x1^2}{(2\,x1^2 + x2^2)^3} - 4\,\frac{1}{(2\,x1^2 + x2^2)^2} & 16\,\frac{x2\,x1}{(2\,x1^2 + x2^2)^3} \\ 16\,\frac{x2\,x1}{(2\,x1^2 + x2^2)^3} & 2 + 8\,\frac{x2^2}{(2\,x1^2 + x2^2)^3} - 2\,\frac{1}{(2\,x1^2 + x2^2)^2} \end{array} \right]$$

Is this matrix positive definite for all (x1,x2)<>(0,0)? It might help to have a common denominator for each term:

> map(normal, evalm(H));

$$\left[\begin{array}{cc} 4\,\frac{8\,x1^6 + 12\,x1^4\,x2^2 + 6\,x1^2\,x2^4 + x2^6 + 6\,x1^2 - x2^2}{(2\,x1^2 + x2^2)^3} & 16\,\frac{x2\,x1}{(2\,x1^2 + x2^2)^3} \\ 16\,\frac{x2\,x1}{(2\,x1^2 + x2^2)^3} & 2\,\frac{8\,x1^6 + 12\,x1^4\,x2^2 + 6\,x1^2\,x2^4 + x2^6 + 3\,x2^2 - 2\,x1^2}{(2\,x1^2 + x2^2)^3} \end{array} \right]$$

>

Eigenvalues will be quite messy

> eigenvals(H):
lambda[1]:="[1];
lambda[2]:=""[2];

$$\lambda_1 := \frac{1}{2}\Big(6\,x2^8 + 48\,x1^2\,x2^6 + 144\,x1^4\,x2^4 + 2\,x2^4 + 24\,x1^2\,x2^2 + 192\,x1^6\,x2^2 + 40\,x1^4 + 96\,x1^8 + 2\,(-72\,x1^2\,x2^{10} + 216\,x1^2\,x2^6 - 10\,x2^{12} + 896\,x1^{12} + 784\,x1^8 + 25\,x2^8 + 1920\,x1^{10}\,x2^2 + 1440\,x1^8\,x2^4 + 320\,x1^6\,x2^6 + 1248\,x1^6\,x2^2 - 120\,x1^4\,x2^8 + 760\,x1^4\,x2^4 + x2^{16} + 256\,x1^{16} + 16\,x2^{14}\,x1^2 + 112\,x2^{12}\,x1^4 + 448\,x2^{10}\,x1^6 + 1120\,x2^8\,x1^8 + 1792\,x1^{10}\,x2^6 + 1792\,x1^{12}\,x2^4 + 1024\,x1^{14}\,x2^2)^{1/2}\Big)\Big/(x2^8 + 8\,x1^2\,x2^6 + 24\,x1^4\,x2^4 + 32\,x1^6\,x2^2 + 16\,x1^8)$$

$$\lambda_2 := \frac{1}{2}\Big(6\,x2^8 + 48\,x1^2\,x2^6 + 144\,x1^4\,x2^4 + 2\,x2^4 + 24\,x1^2\,x2^2 + 192\,x1^6\,x2^2 + 40\,x1^4 + 96\,x1^8 - 2\,(-72\,x1^2\,x2^{10} + 216\,x1^2\,x2^6 - 10\,x2^{12} + 896\,x1^{12} + 784\,x1^8 + 25\,x2^8 + 1920\,x1^{10}\,x2^2 + 1440\,x1^8\,x2^4 + 320\,x1^6\,x2^6 + 1248\,x1^6\,x2^2 - 120\,x1^4\,x2^8 + 760\,x1^4\,x2^4 + x2^{16} + 256\,x1^{16} + 16\,x2^{14}\,x1^2 + 112\,x2^{12}\,x1^4 + 448\,x2^{10}\,x1^6 + 1120\,x2^8\,x1^8 + 1792\,x1^{10}\,x2^6 + 1792\,x1^{12}\,x2^4 + 1024\,x1^{14}\,x2^2)^{1/2}\Big)\Big/(x2^8 + 8\,x1^2\,x2^6 + 24\,x1^4\,x2^4 + 32\,x1^6\,x2^2 + 16\,x1^8)$$

>

The principal minors aren't much better:

> Delta[1] := normal(H[1,1]);

$$\Delta_1 := 4\,\frac{8\,x1^6 + 12\,x1^4\,x2^2 + 6\,x1^2\,x2^4 + x2^6 + 6\,x1^2 - x2^2}{(2\,x1^2 + x2^2)^3}$$

> Delta[2]:=det(H);

$$\Delta_2 := 8\,\frac{x2^8 + 8\,x1^2\,x2^6 + 24\,x1^4\,x2^4 + 2\,x2^4 + 32\,x1^6\,x2^2 + 8\,x1^2\,x2^2 - 3 + 16\,x1^8 + 8\,x1^4}{(2\,x1^2 + x2^2)^4}$$

Can you SEE that these are both strictly positive for all (x1,x2)<>(0,0)? (While it's not a proof, it can sometimes be useful to plot these quantities (the same can be said about the eigenvalues).

> plot3d({Delta[1],Delta[2]}, x1=-2..2, x2=-2..2, view=-4..20, shading=ZHUE, axes=BOXED, style=PATCHCONTOUR);
> plot3d({lambda[1],lambda[2]}, x1=-2..2, x2=-2..2, view=-4..20, shading=ZHUE, axes=BOXED, style=PATCHCONTOUR);

These pictures are misleading because of the singularity at (0,0). Can you think of a way to avoid this?

>

And, finally, what is the domain of the function g? Is this a convex set? What does this have to do with the preceding analysis?

END OF WORKSHEET

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