Solution for §23

#2. $u_t = u_{xx} + u_{yy}$, $u(0,y,t) = u(1,y,t) = 0$, $u(x,0,t) = u(x,1,t) = 0$, $u(x,y,0) = K$.

This fits the basic heat equation in 2 variables with $a = b = 1$ and with zero temperature on all boundaries:

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin(n\pi x) \sin(m\pi x)\, e^{-(n^2\pi^2 + m^2\pi^2)t}$$

To satisfy the IC means:

$$u(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{mn} \sin(n\pi x) \sin(m\pi x) = K.$$

where

$$C_{mn} = \frac{2}{1}\int_0^1 \frac{2}{1}\int_0^a K \sin(n\pi\xi)\,d\xi \,\sin(m\pi\eta)\,d\eta$$

$$= 4K\int_0^1 \left(-\frac{1}{n\pi}\cos(n\pi\xi)\Big|_0^1\right)\sin(m\pi\eta)\,d\eta$$

$$= 4K\int_0^1 \left(-\frac{1}{n\pi}((-1)^n - 1)\right)\sin(m\pi\eta)\,d\eta$$

$$= +\frac{4K}{n\pi}(1-(-1)^n)\int_0^1 \sin(m\pi\eta)\,d\eta$$

$$= \frac{4K}{n\pi}(1-(-1)^n)\,\frac{-1}{m\pi}\cos(m\pi\eta)\Big|_0^1$$

$$= \frac{4K}{n\pi}(1-(-1)^n)\left(-\frac{1}{m\pi}\right)((-1)^m - 1)$$

$$= \frac{4K}{mn\pi^2}(1-(-1)^n)(1-(-1)^m)$$

Solution for §3.1

#7. $u_{tt} = 9u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \sin(x)$, $u_t(x,0) = x$.

This is the 1-d wave equation with ends held at zero.

The solution will be

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\pi}\right)\cos\left(\frac{n\pi 3t}{\pi}\right) + \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{x}\right)\sin\left(\frac{n\pi 3t}{\pi}\right)$$

$$= \sum_{n=1}^{\infty} c_n \sin(nx)\cos(3nt) + \sum_{n=1}^{\infty} d_n \sin(nx)\sin(3nt)$$

where $c_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\,dx = \begin{cases} 1 & n=1 \\ 0 & n\neq 1 \end{cases}$

{ also by inspection of $u(x,0) = \sin x$, $\sum_{n=1}^{\infty} c_n \sin(nx) = \sin x$. }

and

$$3nd_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\left(\frac{-(-1)^n\pi}{n}\right) = \frac{-2(-1)^n}{n}$$

so $d_n = \dfrac{-2(-1)^n}{3n^2}$.

The full solution is:

$$u(x,t) = \sin(x)\cos(3t) + \sum_{n=1}^{\infty}\frac{-2(-1)^n}{3n^2}\sin(nx)\sin(3nt).$$