Solutions for § 2.2

#6. Problem (2.6) is: $u_t = ku_{xx} + F(x,t)$, $u(0,t) = u(L,t) = 0$, $u(x,0) = f(x)$.

The problem we want to solve is the same except the BC are: $u_x(0,t) = u_x(L,t) = 0$.

Generally, we expect this to mean the $\sin\left(\frac{n\pi x}{L}\right)$ are replaced by $\cos\left(\frac{n\pi x}{L}\right)$ — and include the constant term for $n=0$.

To start this process, define a_n to be the Fourier cosine coefficients for the IC:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad (n=0,1,2,\ldots)$$

Because the soln to the corresponding homogeneous problem ($F(x,t)=0$) is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2}$$

We attempt to find a soln to the nonhomogeneous problem as:

$$(*) \qquad u(x,t) = \frac{T_0(t)}{2} + \sum_{n=1}^{\infty} T_n(t)\cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2}$$

To find the $T_n(t)$, first write $F(x,t) = \frac{A_0(t)}{2} + \sum_{n=1}^{\infty} A_n(t)\cos\left(\frac{n\pi x}{L}\right)$.

From $(*)$ we know that $T_n(t)$ is the Fourier cosine coefficient of $u(x,t)$:

$$T_n(t) = \frac{2}{L}\int_0^L u(x,t)\cos\left(\frac{n\pi x}{L}\right)dx$$

Now, differentiate this equation for $T_n(t)$ wrt t:

$$T_n'(t) = \frac{2}{L}\int_0^L u_t \cos\left(\frac{n\pi x}{L}\right)dx \qquad \text{because } u_t = ku_{xx} + F(x,t)$$

$$= \frac{2}{L}\int_0^L \left(ku_{xx} + F(x,t)\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= k\,\frac{2}{L}\int_0^L u_{xx}\cos\left(\frac{n\pi x}{L}\right)dx + \underbrace{\frac{2}{L}\int_0^L F(x,t)\cos\left(\frac{n\pi x}{L}\right)dx}_{A_n(t)}$$

We use integration by parts, twice, to rewrite $\frac{2}{L}\int_0^L u_{xx}\cos\left(\frac{n\pi x}{L}\right)dx$:

$$\frac{2}{L}\int_0^L \underset{dv}{u_{xx}}\,\underset{u}{\cos\left(\frac{n\pi x}{L}\right)}dx = \frac{2}{L}\left(\cancelto{0}{u_x(L,t)}\cos(n\pi) - \cancelto{0}{u_x(0,t)}\cos(0) + \int_0^L u_x\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= \frac{n\pi}{L}\cdot\frac{2}{L}\int_0^L \underset{dv}{u_x}\,\underset{u}{\sin\left(\frac{n\pi x}{L}\right)}dx$$

$$= \frac{n\pi}{L}\cdot\frac{2}{L}\left(u(L,t)\cancelto{0}{\sin(n\pi)} - u(0,t)\cancelto{0}{\sin(0)} - \int_0^L u\cdot\frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)\right)$$

$$= \left(\frac{n\pi}{L}\right)^2 \cdot \frac{-2}{L}\int_0^L u\cos\left(\frac{n\pi x}{L}\right)dx = -\left(\frac{n\pi}{L}\right)^2 T_n(t)$$

This leads to the following ODE for the $T_n(t)$:

$$T_n'(t) = -k\left(\frac{n\pi}{L}\right)^2 T_n(t) + A_n(t)$$

which we write as

$$T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t) = A_n(t).$$

For an initial condition observe that

$$T_n(0) = \frac{2}{L}\int_0^L u(x,0)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = a_n.$$

To solve this 1st order linear IVP we find an integrating factor:

$$\mu(t) = e^{k\left(\frac{n\pi}{L}\right)^2 t}$$

Then

$$\frac{d}{dt}\left(e^{k\left(\frac{n\pi}{L}\right)^2 t} T_n(t)\right) = e^{k\left(\frac{n\pi}{L}\right)^2 t}\left(T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t)\right)$$

$$= e^{k\left(\frac{n\pi}{L}\right)^2 t} A_n(t)$$

Integrate from 0 to t as follows:

$$\int_0^t \frac{d}{dt}\left(e^{k\left(\frac{n\pi}{L}\right)^2 t} T_n(t)\right)dt = \int_0^t e^{k\left(\frac{n\pi}{L}\right)^2 \tau} A_n(\tau)\,d\tau$$

$$e^{k\left(\frac{n\pi}{L}\right)^2 t} T_n(t)\Big|_{t=0}^{t=t} = \int_0^t e^{k\left(\frac{n\pi}{L}\right)^2 \tau} A_n(\tau)\,d\tau$$

$$e^{k\left(\frac{n\pi}{L}\right)^2 t} T_n(t) - \overset{a_n}{\cancel{T_n(0)}} = \int_0^t e^{k\left(\frac{n\pi}{L}\right)^2 \tau} A_n(\tau)\,d\tau$$

so $$T_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(a_n + \int_0^t e^{k\left(\frac{n\pi}{L}\right)^2 \tau} A_n(\tau)\,d\tau\right).$$

Now that we know how to find the $T_n(t)$ we have the full solution as:

$$u(x,t) = \frac{T_0(t)}{2} + \sum_{n=1}^{\infty} T_n(t)\cos\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$