Solutions for §2.1

#3. $u_t = ku_{xx}$, $u_x(0,t) = u_x(4,t) = 0$, $u(x,0) = x^2$.

This is the standard heat equation with $L=4$ and 2 insulated ends.

Separating variables leads to $X'' + \lambda X = 0$, $X'(0) = X'(4) = 0$
$T' + k\lambda T = 0$.

The only nontrivial solutions occur when $\lambda = \left(\frac{n\pi}{4}\right)^2$ and are
$X_n(x) = \cos\left(\frac{n\pi x}{4}\right)$ and $T_n(t) = e^{-k\frac{n^2\pi^2}{16}t}$ for $n = 0, 1, 2, \ldots$

$\lambda = 0$
$X_0(x) = 1$
$T_0(t) = 1$

so $u_n(x,t) = X_n(x)T_n(t) = \cos\left(\frac{n\pi x}{4}\right) e^{-k\frac{n^2\pi^2}{16}t}$

The general soln to the heat eqn & insulated BC is:
$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) e^{-k\frac{n^2\pi^2}{16}t}$$

To match the IC: $u(x,0) = x^2$ we need $x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right)$

so $$a_n = \frac{2}{4}\int_0^4 x^2 \cos\left(\frac{n\pi x}{4}\right) dx = \cdots = \frac{2}{4}\left(\frac{4}{n^2\pi^2}\right)\left(8n\pi x \cos\left(\frac{n\pi x}{4}\right) + (n^2\pi^2x^2 - 32)\sin\left(\frac{n\pi x}{4}\right)\right)\Bigg|_0^4$$

$$= \frac{2}{n^2\pi^2}\left(8n\pi(4)\cos(n\pi) - 0 + (16n^2\pi^2 - 32)\sin(n\pi) - 0\right)$$

$$= \frac{64(-1)^n}{n^2\pi^2} \quad \text{for } n = 1, 2, 3, \ldots$$

For $n=0$: $$a_0 = \frac{2}{4}\int_0^4 x^2\,dx = \frac{1}{2}\frac{1}{3}x^3\Big|_0^4 = \frac{32}{3}.$$

The solution to this problem is
$$u(x,t) = \frac{16}{3} + \sum_{n=1}^{\infty} \frac{64(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{4}\right) e^{-k\frac{n^2\pi^2}{16}t}$$
$$= \frac{16}{3} - \frac{64}{\pi^2}\cos\left(\frac{\pi x}{4}\right) e^{-\frac{k\pi^2}{16}t} + \frac{64}{4\pi^2}\cos\frac{2\pi x}{4} e^{-k\frac{4\pi^2}{16}t} - \cdots$$

#4. $u_t = k u_{xx}$, $u(0,t) = u(2,t) = 0$, $u(x,0) = \sin(\pi x)$

This is the basic heat equation with $L=2$ and 2 ends at zero temperature.

Separating variables leads to $X'' + \lambda X = 0$, $X(0) = X(2) = 0$

$$T' + k\lambda T = 0$$

The only nontrivial solutions occur when $\lambda_n = \left(\frac{n\pi}{2}\right)^2$, with

$$X_n = \sin\left(\frac{n\pi x}{2}\right) \quad \text{and } T_n(t) = e^{-k\left(\frac{n\pi}{2}\right)^2 t}.$$

This gives $u_n(x,t) = \sin\left(\frac{n\pi x}{2}\right) e^{-k\left(\frac{n\pi}{2}\right)^2 t}$

and the general solution is $u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{2}\right) e^{-k\left(\frac{n\pi}{2}\right)^2 t}$

To match the IC requires $\sin(\pi x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{2}\right)$

and so $a_n = \begin{cases} 0 & n \neq 2 \\ 1 & n = 2 \end{cases}$ ← by inspection because $\sin(\pi x)$ can be written as

$0 \sin\left(\frac{\pi x}{2}\right) + 1 \sin\left(\frac{2\pi x}{2}\right) + 0 \sin\left(\frac{3\pi x}{2}\right) + \ldots$

The solution then reduces to a single term:

$$u(x,t) = \sin(\pi x)\, e^{-k\pi^2 t}$$

#6. $u_t = k u_{xx}$, $u(0,t) = 3$, $u(5,t) = \sqrt{7}$, $u(x,0) = x^2$

As this is the heat equation with the temperature held at non-zero values on both ends we first find the linear function that passes through $(0,3)$ and $(5,\sqrt{7})$, namely $\psi(x) = 3 + \frac{\sqrt{7}-3}{5}x$, and write $u(x,t) = v(x,t) + \psi(x)$.

Then $u_t = v_t$ and $u_{xx} = v_{xx} + \psi'' = v_{xx}$ (because $\psi''(x) = 0$). Then

$u_t = k u_{xx} \Rightarrow v_t = k v_{xx}$

$u(0,t) = 3 \Rightarrow v(0,t) + \psi(0) = v(0,t) + 3 = 3 \Rightarrow v(0,t) = 0$

$u(5,t) = \sqrt{7} \Rightarrow v(5,t) + \psi(5) = v(5,t) + \sqrt{7} = \sqrt{7} \Rightarrow v(5,t) = 0$

$u(x,0) = x^2 \Rightarrow v(x,0) + \psi(x) = x^2 \Rightarrow v(x,0) = x^2 - \psi(x) = x^2 - \left(3 + \frac{\sqrt{7}-3}{5}x\right)$.

So v satisfies a heat eqn. with ends held at temperature zero. As such,

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{5}\right) e^{-k\left(\frac{n\pi}{5}\right)^2 t}$$

where the IC (for v) requires: $v(0,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{5}\right) = x^2 - \left(3 + \frac{\sqrt{7}-3}{5}x\right)$

This means the a_n must be Fourier sine coefficients of $x^2 - \left(3 + \frac{\sqrt{7}-3}{5}x\right)$:

$$a_n = \frac{2}{5}\int_0^5 \left(x^2 - \left(3 + \frac{\sqrt{7}-3}{5}x\right)\right) \sin\left(\frac{n\pi x}{5}\right) dx = \ldots = \frac{2}{n\pi}\left((\sqrt{7}-25)(-1)^n - 3\right) - \frac{20}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$= \begin{cases} \frac{2(\sqrt{7}-28)}{n\pi} & n \text{ even} \\ -\frac{2(\sqrt{7}-22)}{n\pi} - \frac{500}{n^3\pi^3} & n \text{ odd.} \end{cases}$$

Then $u(x,t) = 3 + \frac{\sqrt{7}-3}{5}x + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{5}\right) e^{-k\left(\frac{n\pi}{5}\right)^2 t}$

#9. $u_t = 5u_{xx}$, $u(0,t)=0$, $u(4,t)=12$, $u(x,0)=x^2(4-x)$.

This problem ~~also~~ has non-zero temperatures at one end, but the process is the same as if both ends are non-zero. Write $u(x,t) = v(x,t) + 3x$ ← linear fn. through (0,0) & (4,12).

and find $v(x,t)$ that is the solution to:

$$v_t(x,t) = 5v_{xx}(x,t), \quad v(0,t)=v(4,t)=0, \quad v(x,0)=x^2(4-x)-3x$$

This is our basic heat equation with zero temperature at both ends, so

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{4}\right) e^{-5\left(\frac{n\pi}{4}\right)^2 t}$$

To find the a_n, recall the IC (for v):

$$v(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{5}\right) = x^2(4-x)-3x = -x^3+4x^2-3x$$

so $$a_n = \frac{2}{4}\int_0^4 (-x^3+4x^2-3x)\sin\left(\frac{n\pi x}{4}\right)dx = \ldots = \frac{1}{2}\left(\frac{48(-1)^n}{n\pi} - \frac{512\,(2(-1)^n+1)}{n^3\pi^3}\right)$$

$$= \frac{1}{2}\begin{cases} \dfrac{48}{n\pi} - \dfrac{1536}{n^3\pi^3} & n \text{ even} \\[2ex] -\dfrac{48}{n\pi} + \dfrac{512}{n^3\pi^3} & n \text{ odd} \end{cases}$$

Then $$u(x,t) = 3x + v(x,t) = 3x + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{4}\right) e^{-5\left(\frac{n\pi}{4}\right)^2 t}$$

#11. $u_t = 4u_{xx} - 2u_x$, $u(0,t)=u(L,t)=0$, $u(x,0)=1$.

Because of the extra term, look for $u(x,t) = e^{\alpha x+\beta t} v(x,t)$.

Then $u_t = \beta e^{\alpha x+\beta t} v + e^{\alpha x+\beta t} v_t = e^{\alpha x+\beta t}(\beta v + v_t)$

$u_x = e^{\alpha x+\beta t}(\alpha v + v_x)$ and $u_{xx} = e^{\alpha x+\beta t}(\alpha^2 v + 2\alpha v_x + v_{xx})$.

Next, $u_t = 4u_{xx} - 2u_x \Rightarrow \beta v + v_t = 4(\alpha^2 v + 2\alpha v_x + v_{xx}) - 2(\alpha v + v_x)$

$$v_t = 4v_{xx} + (4\alpha^2 - 2\alpha - \beta)v + (8\alpha - 2)v_x$$

simplifies to $\boxed{v_t = 4v_{xx}}$ when $8\alpha - 2 = 0$ and $4\alpha^2 - 2\alpha - \beta = 0$. These conditions are satisfied when $\alpha = \frac{1}{4}$ and $\beta = -1/4$. The new BC are:

$u(0,t) = 0 \Rightarrow e^{\alpha\cdot 0+\beta t} v(0,t) = 0 \Rightarrow \boxed{v(0,t)=0.}$

$u(L,t) = 0 \Rightarrow e^{\alpha\cdot L+\beta t} v(L,t) = 0 \Rightarrow \boxed{v(L,t)=0.}$

The solution to the heat equation with end temperatures held at zero is:

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-4(n\pi/L)^2 t}$$

Almost done. Now $u(x,t) = e^{x/4 - t/4} v(x,t) = e^{\frac{x}{4}-\frac{t}{4}} \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-4\left(\frac{n\pi}{L}\right)^2 t}$

To satisfy the IC (for u): $u(x,0) = e^{x/4}\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = 1$ where

the a_n must be the Fourier sine coefficients of $1/e^{x/4} = e^{-x/4}$.

$$a_n = \frac{2}{L}\int_0^L e^{-x/4}\sin\left(\frac{n\pi x}{L}\right)dx = \ldots = \frac{-16n\pi\,(e^{-L/4}(-1)^n-1)}{16n^2\pi^2 + L^2}$$