§1.3

#1 $X'' + \lambda X = 0$, $X(0) = X'(L) = 0$.

Case 1: $\lambda < 0$ $(\lambda = -\sigma^2)$: $X'' - \sigma^2 X = 0$ $(r^2 - \sigma^2 = 0,\ r = \pm\sigma)$.

$(\sigma > 0)$ $X(x) = c_1 e^{\sigma x} + c_2 e^{-\sigma x}$

$(X'(x) = c_1\sigma e^{\sigma x} - c_2\sigma e^{-\sigma x})$

$X(0) = c_1 + c_2 = 0$ $\quad$ $X'(L) = c_1\sigma e^{\sigma L} - c_2\sigma e^{-\sigma L} = 0$

$c_2 = -c_1$ $\quad$ $\Rightarrow c_1\sigma(e^{\sigma L} + e^{\sigma L}) = 0$

$\therefore \sigma = 0$ but $\sigma > 0$.

so no nontrivial solutions

Case 2: $\lambda = 0$: $X'' = 0$ $(r^2 = 0,\ r = \pm 0)$

$X(x) = c_1 x + c_2$

$(X'(x) = c_1)$

$X(0) = \underline{c_2 = 0}$ $\quad$ $X'(L) = \underline{c_1 = 0}$ $\quad$ $\therefore$ no nontrivial solutions for $\lambda = 0$

Case 3: $\lambda > 0$ $(\lambda = +\sigma^2)$: $X'' + \sigma^2 X = 0$ $(r^2 + \sigma^2 = 0,\ r = \pm i\sigma)$

$(\sigma > 0)$ $X(x) = c_1\cos(\sigma x) + c_2\sin(\sigma x)$

$(X'(x) = -c_1\sigma\sin(\sigma x) + c_2\sigma\cos(\sigma x))$

$X(0) = \underline{c_1 = 0}$ $\quad$ $X'(L) = c_1\sigma\sin(\sigma L) + c_2\sigma\cos(\sigma L) = c_2\sigma\cos(\sigma L) = 0$

To get nontrivial sol'n, choose $\sigma > 0$ so that $\cos(\sigma L) = 0$,

that is $\sigma L = (2k-1)\frac{\pi}{2}$ for $k = 1, 2, \ldots$

so $\sigma = (2k-1)\frac{\pi}{2L}$

Eigenvalues $\lambda_k = +\left(\frac{2k-1}{2}\frac{\pi}{L}\right)^2$

Eigenfunctions: $X_k = \sin\left(\frac{2k-1}{2}x\right)$

#4. $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) + 2X'(L) = 0$.

Case 1: $\lambda < 0 \; (\lambda = -\sigma^2)$ $(\sigma > 0)$: $X'' - \sigma^2 X = 0$

$$X(x) = c_1 e^{\sigma x} + c_2 e^{-\sigma x}$$
$$X'(x) = c_1 \sigma e^{\sigma x} - c_2 \sigma e^{-\sigma x}$$

$X(0) = 0$: $c_1 + c_2 = 0 \;\Rightarrow\; c_2 = -c_1$

$X(L) + 2X'(L) = 0$: $0 = c_1 e^{\sigma L} - c_1 e^{-\sigma L} + 2(c_1 \sigma e^{\sigma L} + c_1 \sigma e^{-\sigma L})$

$$= c_1 \left((1+2\sigma) e^{\sigma L} + (-1+2\sigma) e^{-\sigma L} \right)$$

if $c_1 \neq 0$, then $(1+2\sigma) e^{\sigma L} = (1-2\sigma) e^{-\sigma L}$

but not that the LHS is greater than 1 $(1+2\sigma > 1,\; e^{\sigma L} > 1)$
while the RHS is less than 1 $(1-2\sigma < 1,\; \&\; e^{-\sigma L} < 1)$.

Bottom line: there are no nontrivial solutions with $\lambda < 0$.

Case 2: $\lambda = 0$: $X'' = 0$: $X(x) = c_1 x + c_2$ $\;(X'(x) = c_1)$.

$X(0) = 0$: $c_2 = 0$

$X(L) + 2X'(L) = 0$: $0 = (c_1 L + \cancel{c_2}) + 2c_1 = (L+2)c_1 \Rightarrow c_1 = 0$ (because $L > 0$).

Again, no non-trivial solutions.

Case 3: $\lambda > 0 \; (\lambda = \sigma^2)$ $(\sigma > 0)$: $X'' + \sigma^2 X = 0$: $X(x) = c_1 \cos(\sigma x) + c_2 \sin(\sigma x)$

$\left(X'(x) = -c_1 \sigma \sin(\sigma x) + c_2 \sigma \cos(\sigma x) \right)$

$X(0) = 0$: $\underline{0 = c_1}$ $\;(\text{so } X(x) = c_2 \sin \sigma x,\; X'(x) = c_2 \sigma \cos(\sigma x))$

$X(L) + 2X'(L) = 0$: $0 = c_2 \sin(\sigma L) + 2 c_2 \sigma \cos(\sigma L)$

$$= c_2 \left(\sin(\sigma L) + 2\sigma \cos(\sigma L) \right).$$

Notice that $\sin(\sigma L) + 2\sigma \cos(\sigma L) = 0 \iff \sin(\sigma L) = -2\sigma \cos(\sigma L)$
$\iff \tan(\sigma L) = -2\sigma$.

Now, there are an ∞ # of values of σ for which $\tan(\sigma L) = 2\sigma$.

but we can't write them down explicitly.

For each of these values of σ, σ_n, $(\lambda_n = \sigma_n^2)$ there is a corresponding eigenfunction is

$$X_n(x) = \sin(\sigma_n x).$$