HW Solutions - §9.3

#6.

$x' = x - x^2 - xy = x(1-x-y)$

$y' = 3y - xy - 2y^2 = y(3-x-2y)$

$J = \begin{pmatrix} 1-2x-y & -x \\ -y & 3-x-4y \end{pmatrix}$

(a) critical points: $(0,0)$, $(0,\frac{3}{2})$, $(1,0)$, $(-1,2)$

(b) $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = 3$ so linear system @ $(0,0)$ is an unstable node

& (c) the nonlinear system has an unstable node at $(0,0)$

$J(0,\frac{3}{2}) = \begin{pmatrix} -1/2 & 0 \\ -3/2 & -3 \end{pmatrix} \Rightarrow \lambda_1 = -\frac{1}{2}, \lambda_2 = 3$ (~~saddle point~~ asymptotically stable node)

the nonlinear system has an ~~unstable saddle point~~ asymptotically stable node at $(0, 3/2)$.

$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = 2$ (saddle point)

the nonlinear system has an unstable saddle point at $(1,0)$

$J(-1,2) = \begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix} \Rightarrow \begin{matrix} p = -3 \\ q = -2 \end{matrix}$ (saddle point)

the nonlinear system has an unstable saddle point at $(-1,2)$

#10.

$x' = x + x^2 + y^2$

$y' = y - xy$

$J = \begin{pmatrix} 1+2x & 2y \\ -y & 1-x \end{pmatrix}$

(a) critical points: $(0,0)$, $(-1,0)$

(b) $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda_{1,2} = 1$, $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (unstable proper node)

& (c) the nonlinear system is unstable at $(0,0)$

but could be either a node or a spiral

$J(-1,0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = 2$ (saddle point)

the nonlinear system has an unstable saddle point at $(-1,0)$

#12. $x' = (1+x)\sin y$
$y' = 1 - x - \cos y$

$$J = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix}$$

(a) critical points: $x'=0 \Rightarrow 1+x=0$ or $\sin y = 0$.

$1+x=0 \Rightarrow x=-1$ $\quad\quad$ $\sin y = 0 \Rightarrow y = n\pi \ (n \in \mathbb{Z})$.

when $x=-1$: $y' = 2-\cos y \geq 1$ so no critical points here.

when $y=n\pi$: $y' = 1-x-\cos y = 1-x-(-1)^n = 0 \Rightarrow x = 1-(-1)^n = 1+(-1)^{n+1}$

all critical points are $(1+(-1)^{n+1}, n\pi)$

when n is even these are $(0, n\pi)$ $\quad (n=2k)$

when n is odd these are $(2, n\pi)$. $\quad (n=2k+1)$

(b) n even: $x=0,\ y=n\pi$ so $\cos y = \cos(n\pi) = (-1)^n = 1,\ \sin y = 0$.

&(c)

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad p=0,\ q=0+1=1,\ \Delta = p^2-4q = -4$$
$$(\lambda_{1,2} = \pm i)$$

the linear systems at these critical points are stable centers.

these critical points for the nonlinear system will most likely be spirals but could be unstable or asymptotically stable.

n odd: $x=2,\ y=n\pi$ so $\cos y = \cos(n\pi) = (-1)^n = -1,\ \sin y = 0$.

$$J = \begin{pmatrix} 0 & (1+2)(-1) \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix} \quad \begin{matrix} p=0 \\ q=-3<0 \\ (\lambda_{1,2} = \pm\sqrt{3}) \end{matrix} \quad \text{(saddle points)}$$

these critical points for the nonlinear system will be unstable saddle points.