§7.9 HW Solutions

#2. $\vec{x}' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\vec{x} + \begin{pmatrix} e^t \\ \sqrt{3}e^{-t} \end{pmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) - 3 = \lambda^2 - 4 = (\lambda+2)(\lambda-2).$$

$\lambda = 2$: $(A-2I)\vec{\xi} = \vec{0}$: $\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix}\vec{\xi} = \vec{0}$ $\quad \vec{\xi}^{(1)} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$. $\quad \vec{x}^{(1)} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} e^{2t}$

$\lambda = -2$: $(A+2I)\vec{\xi} = \vec{0}$ $\quad \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\vec{\xi} = \vec{0}$ $\quad \vec{\xi}^{(2)} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$ $\quad \vec{x}^{(2)} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} e^{-2t}$

$\vec{x}_c = \Psi\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $\Psi = \begin{pmatrix} \sqrt{3}e^{2t} & -e^{-2t} \\ e^{2t} & \sqrt{3}e^{-2t} \end{pmatrix}$

$\vec{x}_p = \Psi\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ where $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \Psi^{-1}\begin{pmatrix} e^t \\ \sqrt{3}e^{-t} \end{pmatrix}$

$$= \frac{1}{3+1}\begin{pmatrix} \sqrt{3}e^{-2t} & e^{-2t} \\ -e^{2t} & \sqrt{3}e^{2t} \end{pmatrix}\begin{pmatrix} e^t \\ \sqrt{3}e^{-t} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} \sqrt{3}e^{-t} + \sqrt{3}e^{-3t} \\ -e^{3t} + 3e^t \end{pmatrix}.$$

integrating:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -\sqrt{3}e^{-t} - \frac{1}{\sqrt{3}}e^{-3t} \\ -\frac{1}{3}e^{3t} + 3e^t \end{pmatrix}$$

so $\vec{x}_p = \Psi\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}e^{2t} & -e^{-2t} \\ e^{2t} & \sqrt{3}e^{-2t} \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{3}}{4}e^{-t} - \frac{1}{4\sqrt{3}}e^{-3t} \\ -\frac{1}{12}e^{3t} + \frac{3}{4}e^t \end{pmatrix}$

$$= \begin{pmatrix} \sqrt{3}e^{2t}\left(-\frac{\sqrt{3}}{4}e^{-t} - \frac{1}{4\sqrt{3}}e^{-3t}\right) - e^{-2t}\left(-\frac{1}{12}e^{3t} + \frac{3}{4}e^t\right) \\ e^{2t}\left(-\frac{\sqrt{3}}{4}e^{-t} - \frac{1}{4\sqrt{3}}e^{-3t}\right) + \sqrt{3}e^{-2t}\left(-\frac{1}{12}e^{3t} + \frac{3}{4}e^t\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(-\frac{3}{4} + \frac{1}{12}\right)e^t + \left(-\frac{1}{4} - \frac{3}{4}\right)e^{-t} \\ \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{12}\right)e^t + \left(-\frac{1}{4\sqrt{3}} + \frac{3\sqrt{3}}{4}\right)e^{-t} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}e^t - e^{-t} \\ -\frac{\sqrt{3}}{3}e^t + \frac{2\sqrt{3}}{3}e^{-t} \end{pmatrix}$$

General Solution:

$$\vec{x}(t) = c_1\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}e^{-2t} + \begin{pmatrix} -\frac{2}{3}e^t - e^{-t} \\ -\frac{\sqrt{3}}{3}e^t + \frac{2\sqrt{3}}{3}e^{-t} \end{pmatrix}.$$

#6. $\vec{x}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\vec{x} + \begin{pmatrix} t^{-1} \\ 2t^{-1}+4 \end{pmatrix} \quad (t>0).$

$\det(A-\lambda I) = \det\begin{pmatrix} -4-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix} = (-4-\lambda)(-1-\lambda) - 4 = \lambda^2 + 5\lambda = \lambda(\lambda+5).$

$\lambda = 0$: $(A - 0I)\vec{\xi} = \vec{0}$: $\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\vec{\xi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

$\lambda = -5$: $(A+5I)\vec{\xi} = \vec{0}$ $\quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\vec{\xi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \vec{\xi}^{(2)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\quad \vec{x}^{(2)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-5t}.$

$\vec{x}_c = \Psi \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $\Psi = \begin{pmatrix} 1 & 2e^{-5t} \\ 2 & -e^{-5t} \end{pmatrix}.$

$\vec{x}_p = \Psi \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ where $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \Psi^{-1} \begin{pmatrix} t^{-1} \\ 2t^{-1}+4 \end{pmatrix} =$

$$= \frac{1}{-5e^{-5t}} \begin{pmatrix} -e^{-5t} & -2e^{-5t} \\ -2 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} \\ 2t^{-1}+4 \end{pmatrix}$$

$$= -\frac{1}{5} e^{5t} \begin{pmatrix} -t^{-1}e^{-5t} - 2e^{-5t}(2t^{-1}+4) \\ -2t^{-1} + 2t^{-1} + 4 \end{pmatrix}$$

$$= -\frac{1}{5} e^{5t} \begin{pmatrix} -5t^{-1}e^{-5t} - 8e^{-5t} \\ 4 \end{pmatrix}$$

$$= -\frac{1}{5} \begin{pmatrix} -5t^{-1} - 8 \\ 4e^{5t} \end{pmatrix} = \begin{pmatrix} t^{-1} + 8/5 \\ -\frac{4}{5}e^{5t} \end{pmatrix}.$$

Integrate: $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \ln t + \frac{8}{5}t \\ -\frac{4}{25}e^{5t} \end{pmatrix}$

so $\vec{x}_p = \begin{pmatrix} 1 & 2e^{-5t} \\ 2 & -e^{-5t} \end{pmatrix} \begin{pmatrix} \ln t + \frac{8}{5}t \\ -\frac{4}{25}e^{5t} \end{pmatrix} = \begin{pmatrix} \ln t + \frac{8}{5}t - \frac{8}{25} \\ 2\ln t + \frac{16}{5}t + \frac{4}{25} \end{pmatrix}$

The general solution is

$$\vec{x}(t) = \vec{x}_c + \vec{x}_p = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-5t} + \begin{pmatrix} \ln t + \frac{8}{5}t - \frac{8}{25} \\ 2\ln t + \frac{16}{5}t + \frac{4}{25} \end{pmatrix}.$$