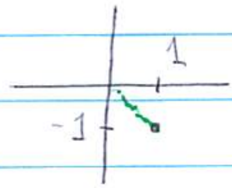
HW Solutions - §4.2

#8. $(1-i)^{1/2}$:



$$1-i = \sqrt{2}\, e^{-\frac{\pi}{4}i} = 2^{1/2} e^{\left(-\frac{\pi}{4}+2m\pi\right)i} \quad \text{for any integer } m.$$

$$(1-i)^{1/2} = \left(2^{1/2} e^{\left(-\frac{\pi}{4}+2m\pi\right)i}\right)^{1/2}$$

$$= 2^{1/4} e^{\frac{1}{2}\left(-\frac{\pi}{4}+2m\pi\right)i}$$

$$= 2^{1/4} e^{\left(-\frac{\pi}{8}+m\pi\right)i}$$

$m=0$: $(1-i)^{1/2} = 2^{1/4} e^{-\pi/8\, i} = 2^{1/4}\cos\left(-\frac{\pi}{8}\right) + i\, 2^{1/4}\sin\left(-\frac{\pi}{8}\right)$

$m=1$: $(1-i)^{1/2} = 2^{1/4} e^{7\pi/8\, i} = 2^{1/4}\cos\left(\frac{7\pi}{8}\right) + i\, 2^{1/4}\sin\left(\frac{7\pi}{8}\right)$

#38. $y^{(4)} - y = 0$. $p_1(t) = 0$.

(a). $W = ce^{-\int 0\, dt} = ce^{-k} = C$ (a constant)

(b) $$W[e^t, e^{-t}, \cos t, \sin t] = \det\begin{bmatrix} e^t & e^{-t} & \cos t & \sin t \\ e^t & -e^{-t} & -\sin t & \cos t \\ e^t & e^{-t} & -\cos t & -\sin t \\ e^t & -e^{-t} & \sin t & -\cos t \end{bmatrix} \quad \left\{\text{expand along first row}\right\}$$

$$= e^t \det\begin{bmatrix} -e^{-t} & -\sin t & \cos t \\ e^{-t} & -\cos t & -\sin t \\ -e^{-t} & \sin t & -\cos t \end{bmatrix} - e^{-t}\det\begin{bmatrix} e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \\ e^t & \sin t & -\cos t \end{bmatrix}$$

$$+ \cos t \det\begin{bmatrix} e^t & -e^{-t} & \cos t \\ e^t & e^{-t} & -\sin t \\ e^t & -e^{-t} & -\cos t \end{bmatrix} - \sin t \det\begin{bmatrix} e^t & -e^{-t} & -\sin t \\ e^t & e^{-t} & -\cos t \\ e^t & -e^{-t} & \sin t \end{bmatrix}$$

$$= e^t\Big(-e^{-t}(\cos^2 t + \sin^2 t) - (-\sin t)(-e^{-t}\cos t - e^{-t}\sin t) + \cos t\,(e^{-t}\sin t - e^{-t}\cos t)\Big)$$

$$- e^{-t}\Big(e^t(\cos^2 t + \sin^2 t) - (-\sin t)(-e^t\cos t + e^t\sin t) + \cos t\,(e^t\sin t + e^t\cos t)\Big)$$

$$+ \cos t\Big(e^t(-e^{-t}\cos t - e^{-t}\sin t) - (-e^{-t})(-e^t\cos t + e^t\sin t) + \cos t\,(-1-1)\Big)$$

$$- \sin t\Big(e^t(e^{-t}\sin t - e^{-t}\cos t) - (-e^{-t})(e^t\sin t + e^t\cos t) - \sin t\,(-1-1)\Big)$$

$$= e^{t}\left(e^{-t}\left(-1-\cancel{\sin t}\cos t-\sin^2 t+\cancel{\sin t}\cos t-\cos^2 t\right)\right)$$
$$-e^{-t}\left(e^{t}\left(1-\sin t\cancel{\cos t}+\sin^2 t+\cancel{\sin t}\cos t+\cos^2 t\right)\right)$$
$$+\cos t\left(-\cos t-\cancel{\sin t}-\cos t+\cancel{\sin t}-2\cos t\right)$$
$$-\sin t\left(\sin t-\cancel{\cos t}+\sin t+\cancel{\cos t}+2\sin t\right)$$
$$=(-1-1)-(1+1)-4\cos^2 t-4\sin^2 t$$
$$=-2-2-4$$
$$=-8.$$

Recall:
$\cosh t=\frac{1}{2}(e^{t}+e^{-t})$
$\sinh t=\frac{1}{2}(e^{t}-e^{-t})$
$\cosh^2 t-\sinh^2 t=1$
$\frac{d}{dt}\cosh t=\sinh t$
$\frac{d}{dt}\sinh t=\cosh t$

(c) $W[\cosh t,\sinh t,\cos t,\sin t]=\det\begin{bmatrix}\cosh t & \sinh t & \cos t & \sin t\\ \sinh t & \cosh t & -\sin t & \cos t\\ \cosh t & \sinh t & -\cos t & -\sin t\\ \sinh t & \cosh t & \sin t & -\cos t\end{bmatrix}$

$$=\cosh t\det\begin{bmatrix}\cosh t & -\sin t & \cos t\\ \sinh t & -\cos t & -\sin t\\ \cosh t & \sin t & -\cos t\end{bmatrix}-\sinh t\det\begin{bmatrix}\sinh t & -\sin t & \cos t\\ \cosh t & -\cos t & -\sin t\\ \sinh t & \sin t & -\cos t\end{bmatrix}$$
$$+\cos t\det\begin{bmatrix}\sinh t & \cosh t & \cos t\\ \cosh t & \sinh t & -\sin t\\ \sinh t & \cosh t & -\cos t\end{bmatrix}-\sin t\det\begin{bmatrix}\sinh t & \cosh t & -\sin t\\ \cosh t & \sinh t & -\cos t\\ \sinh t & \cosh t & \sin t\end{bmatrix}$$

$$=\cosh t\left(\cosh t\left(\cos^2 t+\sin^2 t\right)+\sin t\left(-\sinh t\cos t+\cosh t\sin t\right)+\cos t\left(\sinh t\sin t+\cosh t\cos t\right)\right)$$
$$-\sinh t\left(\sinh t\left(\cos^2 t+\sin^2 t\right)+\sin t\left(-\cosh t\cos t+\sinh t\sin t\right)+\cos t\left(\cosh t\sin t+\sinh t\cos t\right)\right)$$
$$+\cos t\left(\sinh t\left(-\sinh t\cos t+\cosh t\sin t\right)-\cosh t\left(-\cosh t\cos t+\sinh t\sin t\right)+\cos t\left(\cosh^2 t-\sinh^2 t\right)\right)$$
$$-\sin t\left(\sinh t\left(\sinh t\sin t+\cosh t\cos t\right)-\cosh t\left(\cosh t\sin t+\sinh t\cos t\right)-\sin t\left(\cosh^2 t-\sinh^2 t\right)\right)$$

$$=\cosh t\left(\cosh t-\cancel{\sinh t\sin t}\cos t+\cosh t\sin^2 t+\cancel{\sinh t\sin t}\cos t+\cosh t\cos^2 t\right)$$
$$-\sinh t\left(\sinh t-\cancel{\cosh t\sin t}\cos t+\sinh t\sin^2 t+\cancel{\cosh t\sin t}\cos t+\sinh t\cos^2 t\right)$$
$$+\cos t\left(-\sinh^2 t\cos t+\cancel{\cosh t\sinh t}\sin t+\cosh^2 t\cos t-\cancel{\cosh t\sinh t}\sin t+\cos t\right)$$
$$-\sin t\left(\sinh^2 t\sin t+\cancel{\cosh t\sinh t}\cos t-\cosh^2 t\sin t-\cancel{\cosh t\sinh t}\cos t-\sin t\right)$$

$$=\cosh t\,(2\cosh t)-\sinh t\,(2\sinh t)+\cos t\,(2\cos t)-\sin t\,(-2\sin t)$$
$$=2\cosh^2 t-2\sinh^2 t+2\cos^2 t+2\sin^2 t$$
$$=2(1+1)$$
$$=2$$

Now, why did we do all of these calculations?

Because $e^t, e^{-t}, \cos t, \sin t$ are known to be solutions to $y^{(4)} - y = 0$ we used Abel's Theorem to learn that their Wronskian must be a constant. But, what constant?

To determine the constant, all we need is the value of the Wronskian at a single value of t. I don't believe anyone will object if I select $t=0$:

$$W[e^t, e^{-t}, \cos t, \sin t](0) = \det\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

{expand along 4th column}

$$= -0 \cdot \det\left[\quad\right] + 1 \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} - 0 \cdot \det\left[\quad\right] + (-1) \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= +\Big(1(0-1) - 1(0+1) + 1(-1-1)\Big) - \Big(1(1-0) - 1(-1-0) + 1(1+1)\Big) = +(-4) - 4 = -8$$

and

$$W[\cosh t, \sinh t, \cos t, \sin t](0) = \det\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$= 1 \cdot \det\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} - 0 \cdot \det\left[\quad\right] + 1 \cdot \det\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} - 0 \cdot \det\left[\quad\right]$$

$$= 1 \cdot (-1) \cdot (-1-1) + 1\big((-1)(-1-1)\big) = 2 + 2 = 4.$$

One more question: Are you bothered that you found different Wronskians for the different set of fundamental solutions?

You shouldn't worry about this. All we know is that the Wronskian is a constant. The specific constant depends on the specific choice of the solutions and even the order in which you put them in the 1st row. This goes back to Linear Algebra and properties of the determinant.