Homework Solutions - §3.5

#12. $u'' + \omega_0^2 u = \cos \omega_0 t$

Homog: $r^2 + \omega_0^2 = 0 \Rightarrow r = \pm i\omega_0 \Rightarrow y_1 = \cos(\omega_0 t) \quad y_2 = \sin(\omega_0 t)$

Undet. Coef: $Y = t\big(A\cos(\omega_0 t) + B\sin(\omega_0 t)\big) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$

$Y' = A\cos\omega_0 t - A\omega_0 t\sin\omega_0 t + B\sin\omega_0 t + B\omega_0 t\cos\omega_0 t$

$Y'' = -A\omega_0\sin\omega_0 t - A\omega_0\sin\omega_0 t - A\omega_0^2 t\cos\omega_0 t$
$\quad + B\omega_0\cos\omega_0 t + B\omega_0\cos\omega_0 t - B\omega_0^2 t\sin\omega_0 t$

$Y'' + \omega_0^2 Y = -2A\omega_0\sin\omega_0 t - \cancel{A\omega_0^2 t\cos\omega_0 t} + 2B\omega_0\cos\omega_0 t - \cancel{B\omega_0^2 t\sin\omega_0 t}$
$\quad + \omega_0^2\left(\cancel{At\cos\omega_0 t} + \cancel{Bt\sin\omega_0 t}\right)$
$\quad = -2A\omega_0\sin\omega_0 t + 2B\omega_0\cos\omega_0 t = \cos\omega_0 t$

provided $-2A\omega_0 = 0$ and $2B\omega_0 = 1$ so $A = 0$, $B = 1/2\omega_0$.

$\therefore$ particular sol'n is $u_p = \dfrac{t}{2\omega_0}\sin(\omega_0 t)$.

General solution: $u = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \dfrac{t}{2\omega_0}\sin(\omega_0 t)$.

#21. $y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin(3t)$

Homog: $r^2 + 3r = 0 \quad r(r+3) = 0 \Rightarrow r = 0, r = -3 \Rightarrow y_1 = e^{0t} = 1, \ y_2 = e^{-3t}$.

Undet. Coef: $y'' + 3y' = 2t^4 \Rightarrow Y_1 = t(At^4 + Bt^3 + Ct^2 + Dt + E)$ (because 1 is in y_h)

$y'' + 3y' = t^2 e^{-3t} \Rightarrow Y_2 = t(Ft^2 + Gt + H)e^{-3t}$ (because e^{-3t} is in y_h)

$y'' + 3y' = \sin(3t) \Rightarrow Y_3 = J\sin(3t) + K\cos(3t)$ (no terms in y_h)

$\therefore Y_p = At^5 + Bt^4 + Ct^3 + Dt^2 + Et + (Ft^3 + Gt^2 + Ht)e^{-3t} + J\sin(3t) + K\cos(3t)$

From Wolfram Alpha.com: $y(t) = c_1 e^{-3t} + c_2 + \underset{A}{\tfrac{2}{15}t^5} - \underset{B}{\tfrac{2}{9}t^4} + \underset{C}{\tfrac{8}{27}t^3} - \underset{D}{\tfrac{8}{27}t^2} + \underset{E}{\tfrac{16}{81}t} + \left(-\underset{F}{\tfrac{1}{9}t^3} - \underset{G}{\tfrac{1}{9}t^2} - \underset{H}{\tfrac{2}{27}t}\right)e^{-3t}$
$\quad - \underset{J}{\tfrac{1}{18}\sin(3t)} - \underset{K}{\tfrac{1}{18}\cos(3t)}$

(after simplification!)

#25. $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin 2t$

Homog: $r^2 - 4r + 4 = (r-2)^2 = 0 \Rightarrow r = 2$ (mult: 2) $\Rightarrow y_1 = e^{2t} \quad y_2 = te^{2t}$

Undet. Coef: $y'' - 4y' + 4y = 2t^2 \Rightarrow Y_1 = At^2 + Bt + C$

$y'' - 4y' + 4y = 4te^{2t} \Rightarrow Y_2 = t^2(Dt + E)e^{2t}$ (because e^{2t} & te^{2t} are both in y_h)

$y'' - 4y' + 4y = t\sin 2t \Rightarrow Y_3 = (Ft + G)\sin 2t + (Ht + J)\cos 2t$.

$\therefore Y_p = At^2 + Bt + C + (Dt^3 + Et^2)e^{2t} + (Ft + G)\sin(2t) + (Ht + J)\cos(2t)$

From Wolfram Alpha.com: $y = c_1 e^{2t} + c_2 te^{2t} + \underset{D\ (E=0)}{\tfrac{2}{3}t^3 e^{2t}} + \underset{A}{\tfrac{1}{2}t^2} + \underset{B}{t} + \underset{C\ (F=0)}{\tfrac{3}{4}} - \underset{G}{3}\sin(2t) + (\underset{H}{6t} + \underset{J}{3})\cos(2$