Homework Solutions - §3.4

#16. $y'' - y + 0.25y = 0$, $y(0) = 2$, $y'(0) = b$.

$r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2 = 0 \iff r = \frac{1}{2}$ (mult. 2)

$y_1 = e^{t/2}$, $y_2 = te^{t/2}$

$y = c_1 e^{t/2} + c_2 t e^{t/2}$

$y' = \frac{c_1}{2} e^{t/2} + c_2 \left(e^{t/2} + \frac{t}{2} e^{t/2} \right)$

$y(0) = c_1 = 2$

$y'(0) = \frac{c_1}{2} + c_2 = b$

$c_2 = b - \frac{c_1}{2} = b - 1$

$y(t) = 2e^{t/2} + (b-1)te^{t/2}$.

The dominant term in this solution is $te^{t/2}$, so:

$$\lim_{t\to\infty} y(t) = \begin{cases} +\infty & \text{if } b-1 \geq 0 \\ -\infty & \text{if } b-1 < 0 \end{cases}$$

The limiting behavior changes when $b = 1$.

#18. $9y'' + 12y' + 4y = 0$ $y(0) = a > 0$, $y'(0) = -1$.

(a) $9r^2 + 12r + 4 = (3r+2)^2 = 0 \iff r = -\frac{2}{3}$ (mult. 2).

$y_1 = e^{-2t/3}$, $y_2 = te^{-2t/3}$

$y = c_1 e^{-2t/3} + c_2 t e^{-2t/3}$

$y' = -\frac{2c_1}{3} e^{-2t/3} + c_2 \left(e^{-2t/3} - \frac{2t}{3} e^{-2t/3} \right)$

$y(0) = c_1 = a$

$y'(0) = -\frac{2c_1}{3} + c_2 = -1$

$c_2 = -1 + \frac{2c_1}{3} = -1 + \frac{2a}{3}$.

$y(t) = a e^{-2t/3} + \left(\frac{2a}{3} - 1\right) te^{-2t/3}$

(b) Since $y(0) = a > 0$, all solutions start positive.

If $\frac{2a}{3} - 1 \geq 0$ then both terms are positive and so the solution is always positi

If $\frac{2a}{3} - 1 < 0$ then $\lim_{t\to\infty} y(t) = -\infty$, so these solutions become negative.

This separation occurs when $\frac{2a}{3} - 1 = 0$, that is for $a = \frac{3}{2}$.