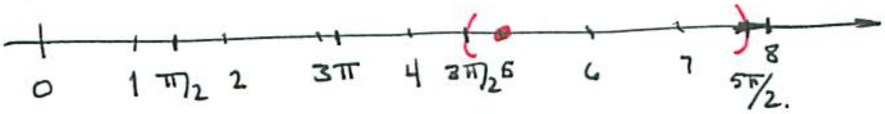
Exam 2 - Key

1. standard form: $y''' + \frac{8}{t} y'' + \frac{\tan t}{t^2} y = \frac{\cos t}{t^2}$

(a) This DE is of order 3 and it's linear.
It does not have constant coefficients (because of the $\frac{8}{t}$ and $\frac{\tan t}{t^2}$).
We've not seen any method to solve the corresponding homogeneous DE.

(b). The coefficients are discontinuous at $t=0$ and every odd multiple of $\pi/2$.



The solution is guaranteed to exist on the open interval $(3\pi/2, 5\pi/2)$.

(c) By Abel's Theorem, $W[y_1, y_2, y_3] = C e^{-\int \frac{8}{t} dt} = C e^{-8 \ln t} = \frac{C}{t^8}$.

2. $y'' + 2y' + 5y = 0 \quad y(0) = 2, \ y'(0) = 4$

$r^2 + 2r + 5 = 0 \qquad r = \frac{1}{2}\left(-2 \pm \sqrt{2^2 - 5 \cdot 4}\right) = \frac{1}{2}(-2 \pm 4i) = -1 \pm 2i.$

Gen'l sol'n: $y = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$

To solve the IVP: $y' = -c_1 e^{-t} \cos 2t - 2c_1 e^{-t} \sin 2t - c_2 e^{-t} \sin 2t + 2c_2 e^{-t} \cos 2t$

so $y(0) = C_1 = 2$
$y'(0) = -C_1 + 2C_2 = 4$ so $2c_2 = 4 + 2 = 6 \longrightarrow c_2 = 3.$

$\therefore y = 2e^{-t} \cos 2t + 3e^{-t} \sin 2t$

3. $y'' - 6y' + 9y = \frac{e^{3t}}{t}$ (a) homogeneous: $y'' - 6y' + 9y = 0$
$r^2 - 6r + 9 = (r-3)^2 = 0 \therefore r = 3$ (mult: 2)
$y_h = c_1 e^{3t} + c_2 t e^{3t}.$

(b) $y_p = u_1 e^{3t} + u_2 t e^{3t}$ where
$\cancel{e^{3t}} u_1' + t \cancel{e^{3t}} u_2' = 0$
$3\cancel{e^{3t}} u_1' + (3t+1)\cancel{e^{3t}} u_2' = \frac{\cancel{e^{3t}}}{t}$

$u_1' + t u_2' = 0 \ (-3)$
$3u_1' + (3t+1)u_2' = 1/t \ +$
$u_2' = 1/t$
$u_1' = -1$

Thus $u_2 = \ln t$ and $u_1 = -t$, which gives:
$y_p = \underbrace{-te^{3t}}_{\text{in } y_h \text{, so can be omitted.}} + \ln t \cdot t e^{3t}$

Gen'l sol'n: $y = c_1 e^{3t} + c_2 t e^{3t} + t(\ln t) e^{3t}.$

4. $2y^{(4)} + y''' - 9y'' + 4y' + 4y = 0 \quad y(0) = y''(0) = 2, \; y'(0) = y'''(0) = 0$

$$2r^4 - r^3 - 9r^2 + 4r + 4 = (r-1)(2r^3 + r^2 - 8r - 4)$$
$$= (r-1)(r-2)(2r^2+3r+2)$$
$$= (r-1)(r-2)(r+2)(2r+1)$$

$r = 1$
$r = 2$
$r = -2$
$r = -1/2$.

Gen'l sol'n: $y = c_1 e^t + c_2 e^{2t} + c_3 e^{-2t} + c_4 e^{-t/2}$

$$y' = c_1 e^t + 2c_2 e^{2t} - 2c_3 e^{-2t} - \frac{c_4}{2} e^{-t/2}$$
$$y'' = c_1 e^t + 4c_2 e^{2t} + 4c_3 e^{-2t} + \frac{c_4}{4} e^{-t/2}$$
$$y''' = c_1 e^t + 8c_2 e^{2t} - 8c_3 e^{-2t} - \frac{c_4}{8} e^{-t/2}$$

if you write the 4 sol'ns in a different order, your system will have the columns in a different order, so you must change the order of the coeff's.

Plug in $t=0$:
$$y(0) = c_1 + c_2 + c_3 + c_4 = 2$$
$$y'(0) = c_1 + 2c_2 - 2c_3 - \frac{c_4}{2} = 0$$
$$y''(0) = c_1 + 4c_2 + 4c_3 + \frac{c_4}{4} = 2$$
$$y'''(0) = c_1 + 8c_2 - 8c_3 - \frac{c_4}{8} = 0$$

Sol'n: $c_1 = 2/3$
$c_2 = 1/10$
$c_3 = 1/6$
$c_4 = 16/15$

Sol'n to IVP: $\underline{y = \frac{2}{3} e^t + \frac{1}{10} e^{2t} + \frac{1}{6} e^{-2t} + \frac{16}{15} e^{-t/2}}$.

5. $y^{(4)} + 4y'' = 3\sin t + 10te^t + 8$.

Homogeneous: $r^4 + 4r^2 = r^2(r^2+4) = 0 \quad r = 0$ (mult: 2), $r = \pm 2i$

$$y_h = c_1 + c_2 t + c_3 \cos 2t + c_4 \sin 2t.$$

Undetermined Coeff:

$$Y = \underbrace{A\cos t + B\sin t}_{3\sin t} + \underbrace{(Ct+D)e^t}_{10te^t} + \underbrace{Et^2}_{8}$$

because 1 & t are in homog. sol'n.

Extra Credit: $Y' = -A\sin t + B\cos t + (Ct+D+C)e^t + 2Et$

$$Y'' = -A\cos t - B\sin t + (Ct+D+2C)e^t + 2E$$
$$Y''' = A\sin t - B\cos t + (Ct+D+3C)e^t$$
$$Y^{(4)} = A\cos t + B\sin t + (Ct+D+4C)e^t$$

Now $Y^{(4)} + 4Y'' = A\cos t + B\sin t + (Ct+D+4C)e^t$
$$-4A\cos t - 4B\sin t + 4(Ct+D+2C)e^t + 8E$$
$$= -3A\cos t - 3B\sin t + (5Ct + 5D + 12C)e^t + 8E = 3\sin t + 10te^t + 8$$

so $-3A = 3 \quad 5C = 10 \quad 8E = 8$
$-3B = 0 \quad 5D + 12C = 0$

These give: $A = -1 \quad C = 2 \quad E = 1$
$B = 0 \quad D = -\frac{24}{5}$

so $\underline{Y = -\cos t + (2t - \frac{24}{5})e^t + t^2}$.