Worked Example
(3x3 w/ complex roots)

§7.6 #8. $\vec{x}' = A\vec{x}$ with $A = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}$.

Step 1: Find e-values:

$$\det(A-\lambda I) = \det\begin{pmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{pmatrix} = (-3-\lambda)\big((-1-\lambda)(-\lambda)+0\big) - 0 + 2\big(1(-1)+2(-1-\lambda)\big)$$

$$= -(3+\lambda)(1+\lambda)\lambda + 2(-3-2\lambda)$$

$$= -(3+4\lambda+\lambda^2)\lambda - 6 - 4\lambda$$

$$= -\lambda^3 - 4\lambda^2 - 7\lambda - 6$$

integer factor must be a factor of -6

$$= -(\lambda+2)(\lambda^2+2\lambda+3)$$

3 e-values, all different, but 2 are complex conjugates.

so $\lambda_1 = -2$

$\lambda_{2,3} = \frac{1}{2}\left(-2 \pm \sqrt{4+12}\right) = \frac{1}{2}\left(-2 \pm 2\sqrt{2}\,i\right) = -1 \pm \sqrt{2}\,i.$

Step 2: Find eigenvectors for each eigenvalue.

$\lambda_1 = -2$: $(A+2I)\vec{x} = \vec{0}$:

$$\begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix} \xrightarrow[\text{③}-2\text{①}\to\text{③}]{\text{②}+\text{①}\to\text{②}} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix}$$

$$\xrightarrow{\text{③}+\text{②}\to\text{③}} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$-x_1 + 2x_3 = 0$ $\quad$ $x_1 = 2x_3$

$x_2 + 2x_3 = 0$ $\quad$ $x_2 = -2x_3$

$x_3 = x_3$

$$\vec{x} = \begin{pmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\therefore \vec{v}^{(1)} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\vec{x}^{(1)}(t) = e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

$$\lambda_3 = -1-\sqrt{2}\,i: \quad A + (1+\sqrt{2}i)I = \begin{pmatrix} -2+\sqrt{2}i & 0 & 2 \\ 1 & \sqrt{2}i & 0 \\ -2 & -1 & 1+\sqrt{2}i \end{pmatrix}$$

$$\xrightarrow{① \leftrightarrow ②} \begin{pmatrix} 1 & \sqrt{2}\,i & 0 \\ -2+\sqrt{2}i & 0 & 2 \\ -2 & -1 & 1+\sqrt{2}i \end{pmatrix} \xrightarrow[③+2① \to ③]{②+(2-\sqrt{2}i)① \to ②} \begin{pmatrix} 1 & \sqrt{2}\,i & 0 \\ 0 & 2+2\sqrt{2}i & 2 \\ 0 & -1+2\sqrt{2}i & 1+\sqrt{2}i \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}② \to ②} \begin{pmatrix} 1 & \sqrt{2}\,i & 0 \\ 0 & 1+\sqrt{2}i & 1 \\ 0 & -1+2\sqrt{2}i & 1+\sqrt{2}i \end{pmatrix} \xrightarrow{③-(1+\sqrt{2}i)② \to ③} \begin{pmatrix} 1 & \sqrt{2}\,i & 0 \\ 0 & 1+\sqrt{2}i & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-1+2\sqrt{2}i - (1+\sqrt{2}i)(1+\sqrt{2}i)$$
$$= -1+2\sqrt{2}i - (1+2\sqrt{2}i-2) = -1+2\sqrt{2}i - (-1+2\sqrt{2}i) = 0.$$

so
$$x_1 + \sqrt{2}i\,x_2 = 0$$
$$(1+\sqrt{2}i)x_2 + x_3 = 0$$
$$\Longrightarrow \quad \begin{aligned} x_1 &= -\sqrt{2}i\,x_2 \\ x_2 &= x_2 \\ x_3 &= -(1+\sqrt{2}i)x_2 \end{aligned} \qquad \bar{x} = \begin{pmatrix} -\sqrt{2}i\,x_2 \\ x_2 \\ -(1+\sqrt{2}i)x_2 \end{pmatrix} = x_2 \begin{pmatrix} -\sqrt{2}i \\ 1 \\ -1-\sqrt{2}i \end{pmatrix}$$

To find 2 real-valued solutions:

$$e^{-\lambda t}\,\bar{v} = e^{(-1-\sqrt{2}i)t} \begin{pmatrix} -\sqrt{2}\,i \\ 1 \\ -1-\sqrt{2}\,i \end{pmatrix}$$

$$= e^{-t}\left(\cos(\sqrt{2}t) - i\sin(\sqrt{2}t)\right)\left(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} -\sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix}\right)$$

$$= e^{-t}\left(\cos(\sqrt{2}t)\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \sin(\sqrt{2}t)\begin{pmatrix} -\sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix}\right)$$

$$+ i e^{-t}\left(\cos(\sqrt{2}t)\begin{pmatrix} -\sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} - \sin(\sqrt{2}t)\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right)$$

$$= \underbrace{e^{-t}\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix}}_{\vec{x}^{(2)}} + i\,\underbrace{e^{-t}\begin{pmatrix} -\sqrt{2}\cos(\sqrt{2}t) \\ -\sin(\sqrt{2}t) \\ -\sqrt{2}\cos(\sqrt{2}t) + \sin(\sqrt{2}t) \end{pmatrix}}_{\vec{x}^{(3)}}$$

Note that because the real e-value is negative and the real part of the ~~[illegible]~~ complex-valued e-values are negative, all solutions approach the origin as $t \to \infty$.