MATH 520 (Section 501)
Prof. Meade

University of South Carolina
Fall 2006

Exam 1
September 22, 2006

Name: Key
SS # (last 4 digits): ____________

Instructions:

1. There are a total of 4 problems on 5 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
5. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	24	
2	30	
3	28	
4	18	
Total	100	

Good Luck!

1. (24 points) Eight differential equations and four slope fields are given below.

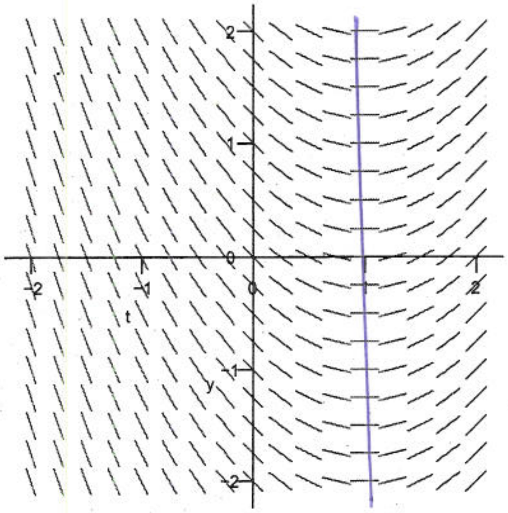
   (a) [4 points] Find the zero isoclines for differential equation (i).

   $\frac{dy}{dt} = 0 \iff 1 - y^2 = 0 \iff y^2 = 1 \iff \underline{y = \pm 1}$

   (b) [4 points] Draw the zero isoclines on slope field (A). ($t = 1$)

   (c) [16 points] Determine the equation that coresponds to each slope field and state briefly how you know your choice is correct.

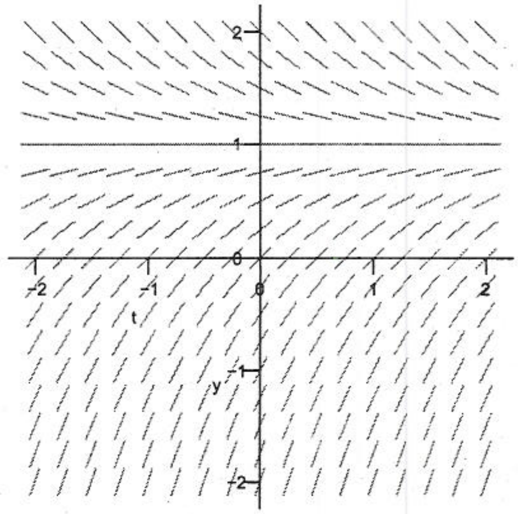
   (i) $y' = 1 - y^2$ (ii) $y' = t - 1$ (iii) $y' = 1 - y$ (iv) $y' = 1 - t$
   (v) $y' = y^2 - t^2$ (vi) $y' = t^2 - y^2$ (vii) $y' = 1 + y$ (viii) $y' = y^2 - 1$



(A) (ii)

nullcline: $t = 1$

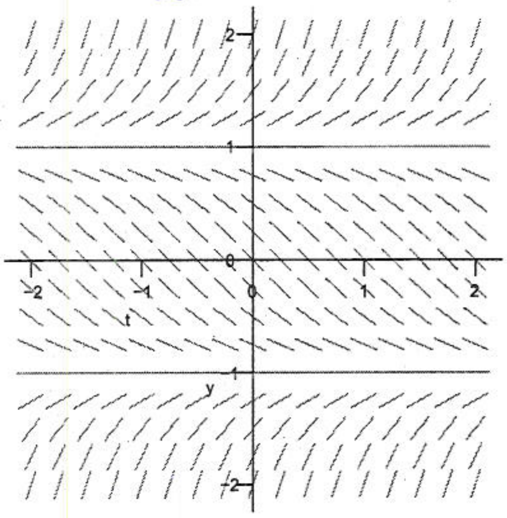
when $t = 0$: $\frac{dy}{dt} < 1$



(B) (iii)

nullclines $y = 1$

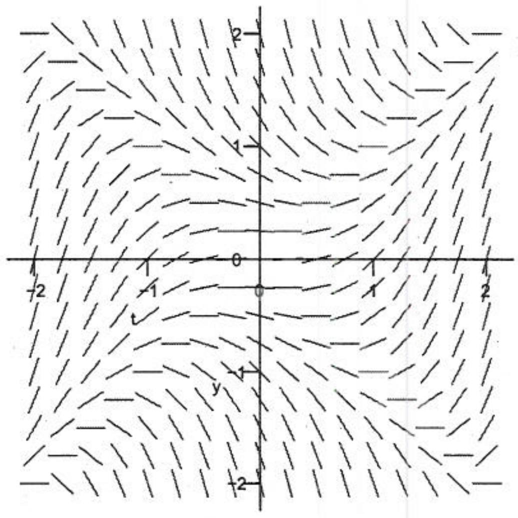
when $y = 0$: $\frac{dy}{dt} > 0$



(C) (viii)

nullclines : $y = \pm 1$

when $y = 0$: $\frac{dy}{dt} < 0$



(D) (vi)

nullclines : $y = \pm t$

when $y = 0$: $\frac{dy}{dt} \geq 0$.

2. (30 points) A model for the vertical flight of a projectile launched from earth from the ground with velocity V in the absence of air resistance is

$$\frac{d^2z}{dt^2} = -\frac{gR^2}{(R+z)^2}.$$

(a) [3 points] What is the order of this differential equation? (Explain.)

2nd order $\left(\frac{d^2z}{dt^2}\right)$

(b) [3 points] Is this differential equation linear or nonlinear? (Explain.)

nonlinear; RHS has z in denominator & also has z^2.

(c) [6 points] What initial conditions complete the description of this situation?

$z(0) = 0$ (on ground)

$\frac{dz}{dt}(0) = V$ (launched w/ velocity V)

(d) [12 points] Nondimensionalize the initial value problem using R as the reference length. For the reference time, use the time required to travel the distance R at the constant velocity V.

NOTE: *Do not forget to nondimensionalize the initial conditions.*

Let $u = \frac{z}{R}$ $\quad VT = R \Rightarrow T = R/V.$

$\tau = \frac{t}{T} = \frac{tV}{R}$

Define $u(\tau) = \frac{z(t)}{R}$, so $z(t) = R\,u(\tau)$.

Then $\frac{dz}{dt} = R\frac{du}{d\tau}\frac{d\tau}{dt} = V\frac{du}{d\tau}$

$\frac{d^2z}{dt^2} = \frac{d}{dt}\left(V\frac{du}{d\tau}\right) = V\frac{d}{d\tau}\left(\frac{du}{d\tau}\right)\frac{d\tau}{dt} = V\frac{d^2u}{d\tau^2}\left(\frac{V}{R}\right) = \frac{V^2}{R}\frac{d^2u}{d\tau^2}$

Thus: $\frac{d^2z}{dt^2} = -\frac{gR^2}{(R+z)^2} = -\frac{gR^2}{(R+Ru)^2} = -\frac{gR^2}{R^2(1+u)^2} = \frac{-g}{1+u^2}$

$z(0) = 0 \to \boxed{u(0) = 0}$

$\frac{dz}{dt}(0) = V \Rightarrow V\frac{du}{d\tau}(0) = V$

$\boxed{\frac{du}{d\tau}(0) = 1}$

Combining:

$\frac{V^2}{R}\frac{d^2u}{d\tau^2} = -\frac{g}{(1+u)^2}$

so that

$\frac{d^2u}{d\tau^2} = -\frac{gR}{V^2}\frac{1}{(1+u)^2}$

$= \frac{\alpha}{(1+u)^2}$

where $\alpha = -gR/V^2$.

(e) [6 points] Your dimensionless model should include one dimensionless parameter. Express this parameter in terms of g, R, and V. Verify that the dimensionless parameter is, in fact, dimensionless.

$\alpha = -\frac{gR}{V^2}$ $\quad [\alpha] = \left[-\frac{gR}{V^2}\right] = \frac{m/s^2 \cdot m}{(m/s)^2} = 1$ ✓

3. (28 points) Consider the initial value problem

$$(y-1)\frac{dy}{dt} = t, \qquad y(0) = 1.$$

(a) [8 points] What does the Existence and Uniqueness Theorem say about solutions to this initial value problem?

$$\frac{dy}{dt} = \frac{t}{y-1} \qquad y(0)=1.$$

The Existence & Uniqueness Theorem does not apply when $y=1$.

(b) [15 points] Try to solve the initial-value problem.

$$\int (y-1)\frac{dy}{dt}dt = \int t\, dt$$

$$\frac{1}{2}y^2 - y = \frac{1}{2}t^2 + C.$$

$y=1, t=0$: $\frac{1}{2} - 1 = C \iff C = -1/2.$

$$\frac{1}{2}y^2 - y = \frac{1}{2}t^2 - \frac{1}{2}.$$

$$y^2 - 2y = t^2 - 1.$$

To solve for y: 1. use quadratic formula: $y^2 - 2y + (1-t^2) = 0.$

$$y = \frac{1}{2}\left(-(-2) \pm \sqrt{4 - 4(1)(1-t^2)}\right)$$

$$= \frac{1}{2}\left(2 \pm \sqrt{4t^2}\right)$$

$$\cancel{= \frac{1}{2}(2 \pm 2|t|)} \quad \frac{1}{2}(2 \pm 2t) = 1 \pm t.$$

or 2. complete the square

$$y^2 - 2y + 1 = t^2 \qquad |y-1| = |t| \qquad \Rightarrow y = 1 \pm t.$$

$$(y-1)^2 = t^2 \qquad y-1 = \pm t$$

(c) [5 points] Are your results in (a) and (b) consistent? (Explain.)

Yes. The fact that there are 2 solutions in (b) is possible because the result in (a) has no information about the solutions.

4. (18 points) [6 points each] Consider the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$$

(a) [6 points] What is the guaranteed interval of existence for the solution to this differential equation with initial conditions $y(0) = 1$ and $y'(0) = 0$?

The coefficients of y' and y are continuous for all $1-x^2 \neq 0$, i.e. $x \neq \pm 1$. Because the IC is given at $x_0 = 0$, the interval of existence for the solution will be $-1 < x < 1$.

(b) [6 points] Are there initial conditions for which a unique solution is not guaranteed? If so, give one such set of initial conditions.

Yes. The theorem will not apply when IC are specified at $x = 1$ or $x = -1$. One such set of IC is: $y(1) = 0$, $y'(1) = 0$.

(c) [2 points] For each positive integer n there is a polynomial of degree n that satisfies the equation with $y(1) = 1$. Find the polynomial solution with degree $n = 0$ to the differential equation with $y(1) = 1$.

poly. degree $n = 0 \Rightarrow y = C$: $y' = 0$, $y'' = 0$.

$$(1-x^2)0 - 2x0 + 0(1)C = 0$$
$$0 = 0 \text{ (no constraint)}$$

$y(1) = \underline{\underline{C = 1}}$ $\quad \therefore y(x) = 1.$

(d) [2 points] Repeat (c) for $n = 1$.

poly degree $n = 1 \Rightarrow y = mx + b$: $y' = m$, $y'' = 0$.

$$(1-x^2)0 - 2x(m) + 1(2)(mx+b) = 0$$
$$-2mx + 2mx + 2b = 0$$
$$2b = 0 \Rightarrow b = 0.$$

$y(1) = m + b = 1 \Rightarrow m = 1.$ $\quad \therefore y(x) = x.$

(e) [2 points] Are the results in (c) and (d) consistent with your answers in (a) and (b)? (Explain.)

Yes. The existence & uniqueness theorem tells us nothing about the case when the IC is given at $x = 1$. So, anything that happens is consistent with this.