Chapter 15 Solutions

#15.2 (a) "$\exists$" in conclusion suggests using the construction method.

(b) Recall that the def'n of continuity involves a "$\forall$". ~~Using~~ Use the forward-backward method.

(c) Note that $a = \pm b$ means $a = +b$ or $a = -b$; this ~~statement~~ statement would be proven using the "either-or" method (which we did not discuss).

(d) With the "$\nexists$" in the conclusion, this would seem to be a good time to use contradiction.

(e) With a "$\forall$" (first) in the conclusion this proof would begin with the choose method.

#15.4 (a) Assume: $p > 1$, not prime
$\Rightarrow \exists k$ (integer) s.t. $k \mid p$.

Conclude: $\exists$ integer m with $1 \le m \le \sqrt{p}$ s.t. $m \mid p$.

How: Forward-Backward method, including the construction of k.

(b) Assume: f, g are continuous at p.
$\Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x - p| < \delta$ & $|f(x) - f(p)| < \varepsilon$

Conclude: $f + g$ is continuous at p
$\Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x - p| < \delta$ & $|f(x) - f(p)| < \varepsilon$.

How: Start with choose method, then bring in specialization to ~~get~~ apply assumption to the particular x chosen by the choose method.

(c) Assume: $a \mid b$ and $b \mid a$

Conclude: $a = \pm b$; ~~that~~ that is $a = +b$ or $a = -b$.
How: Use either/or method to consider both options.

(d) Assume: 1) $\forall\ x \in \mathbb{R},\ f(x) \le g(x)$.

2) $\nexists\ M > 0$ s.t. $f(x) \le M$

~~Conclude~~: 3) $\exists\ M > 0$ s.t. $g(x) \le M$.

Conclude: nothing, will work forward to detect contradiction.

(e) Assume: f, g continuous at x

Conclude: $\forall$ real $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$\forall$ real y with $|x-y| < \delta \Rightarrow \left| f(x) + g(x) - (f(y) - g(y)) \right| < \varepsilon$.

#13.6: Let S, T be sets with $S \subseteq T$.

(a) Choose method: Let $x \in S$ be given

:

$x \le \Sigma$

So $\forall\ x \in S,\ x \le \Sigma$

(b) Specialization: By hypothesis, $\forall x \in T,\ x \le \Sigma$.

Specialize to those elements of T that are also in S.

:

(c) Contradiction: Assume $\forall x \in T,\ x \le \Sigma$ and ~~$\forall$~~ $\exists\ x \in S$ s.t. $x > \Sigma$.

Contradiction is that because $S \subseteq T$, $x \in S \Rightarrow x \in T$

so both $x \le \Sigma$ and $x > \Sigma$. Not possible!

(d) Contrapositive: Assume $\exists\ x \in S$ s.t. $x > \Sigma$

and prove that $\exists x \in T$ s.t. $x > \Sigma$. (by contradiction)

(This is immediately true because the $x \in$ ~~S~~ is also

in T, so the same x works for both parts.)

#15.B. Claim: if ABC is a right triangle with sides of integer length a and b and hypotenuse of integer length c, then $\frac{1}{2}ax^2+cx+b$ has a rational root.

H: ABC is a right triangle with integer length sides a, b, c (with c the hypotenuse)

A1. $a^2+b^2=c^2$ (prop. of right triangle)

A2. roots of $\frac{a}{2}x^2+cx+b$ are

$$x = \frac{-c \pm \sqrt{c^2-4(\frac{a}{2})b}}{2(\frac{a}{2})} = \frac{-c \pm \sqrt{c^2-2ab}}{a}$$

$$= \frac{-c \pm \sqrt{a^2+b^2-2ab}}{a} = \frac{-c \pm \sqrt{a^2-2ab+b^2}}{a}$$

$$= \frac{-c \pm \sqrt{(a-b)^2}}{a} = \frac{-c \pm |a-b|}{a}$$

A3. $-c \pm |a-b|$ ~~are~~ are integers

A4. a is an integer (and not 0 - because it's a length of a leg of a triangle).

B1 $x = \frac{p}{q}$ for integers p & q

B. The roots of $\frac{a}{2}x^2+cx+b$ are rational numbers.

Proof. Let a & b be the (positive) lengths of the legs of the right triangle ABC & let c be the integer length of its hypotenuse. Note that $c^2=a^2+b^2$. The roots of $\frac{1}{2}ax^2+cx+b$

are $x = \frac{-c \pm \sqrt{c^2-4(\frac{a}{2})b}}{2(a/2)} = \cdots = \frac{-c \pm |a-b|}{a}$.

Since $-c \pm |a-b|$ is an integer and a is a positive integer, these roots can be written in the form $x=\frac{p}{q}$ where $p = -c \pm |a-b|$ and $q=a$ are both integers. So ~~are~~ both roots of $\frac{a}{2}x^2+cx+b$ are rational numbers. ∎