# Chapter 12 Solutions

#12.3 (b) Induction cannot be used for statements of the form "$\forall$ real numbers ..." because there is no "next" real number after a given real number.

When we assume $P(n)$ is true and prove that $P(n+1)$ is true we are showing it's possible to take the next step. This makes sense for integers, but not for real numbers.

#12.8 Claim: $\forall k \geq 1$ (integers), $\sum_{i=0}^{k-1} 2^i = 2^k - 1$.

Proof (by induction): Let $P(k)$ be the statement that $\sum_{i=0}^{k-1} 2^i = 2^k - 1$.

Step 1: Let $k=1$. Then $\sum_{i=0}^{1-1} 2^i = 2^0 = 1$ and $2^k - 1 = 2^1 - 1 = 2 - 1 = 1$.

so $P(1)$ is true.

Step 2: Assume $P(n)$ is true, i.e. $\sum_{i=0}^{n-1} 2^i = 2^n - 1$.

Let's look at $P(n+1)$: is $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$?

$$
\begin{aligned}
\sum_{i=0}^{n} 2^i &= \sum_{i=0}^{n-1} 2^i + 2^n \\
&= (2^n - 1) + 2^n \qquad \text{by } P(n) \\
&= 2 \cdot 2^n - 1 \\
&= 2^{n+1} - 1. \qquad \text{so } P(n+1) \text{ is true.}
\end{aligned}
$$

~~Since~~ By the Principle of Mathematical Induction, $P(k)$ is true for all integers $k \geq 1$. ∎

#12.12 Claim: Every set with $n$ elements has $2^n$ subsets.

Proof (by induction):

For each positive integer $n \geq 1$, let $P(n)$ be the statement that a set with $n$ elements has $2^n$ subsets.

Step 1: Let $n=1$: A set with one element is $\{e\}$.
It has 2 subsets $\{e\}$ and $\{\}=\phi$; $2^1=2$. so $P(1)$ is true.

Step 2: Suppose $P(n)$ is true.
Consider a set with $n+1$ elements. There are $2^n$ subsets that do not use the $n+1^{st}$ element. There are another $2^n$ subsets of the $1^{st}$ $n$ elements that also have the $(n+1)^{st}$ element.
The total number of subsets is $2^n+2^n=2^n \cdot 2=2^{n+1}$;
so $P(n+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for every $n \geq 1$. That is, every set of $n$ elements has $2^n$ subsets. $\boxtimes$

Example: $n=2$: subsets of $\{a,b\}$ are $\{a,b\}$, $\{a\}$, $\{b\}$, $\{\}=\phi$.
$n=3$: subsets of $\{a,b,c\}$ are $\{a,b\}$, $\{a\}$, $\{b\}$, $\{\}$
and $\{a,b,c\}$, $\{a,c\}$, $\{b,c\}$, $\{c\}$

#12.17 Claim: $(\cos(x)+i\sin(x))^n = \cos(nx)+i\sin(nx)$ for every integer $n \geq 1$.

Proof: (by induction)

Let $P(n)$ be the statement that $(\cos(x)+i\sin(x))^n = \cos(nx)+i\sin(nx)$.

Step 1: Let $n=1$. $(\cos(x)+i\sin(x))^1 = \cos(1x)+i\sin(1x)$ so $P(1)$ is true.

Step 2: Suppose $P(n-1)$ is true (for $n \geq 2$).

That is $(\cos(x)+i\sin(x))^{n-1} = \cos((n-1)x)+i\sin((n-1)x)$.

Then to show $P(n)$ is true:

$$
\begin{aligned}
(\cos(x)+i\sin(x))^n &= \underline{(\cos(x)+i\sin(x))^{n-1}}(\cos(x)+i\sin(x)) \\
\left(\text{because } P(n-1) \text{ is true}\right) \quad &= \underline{(\cos((n-1)x)+i\sin((n-1)x))}(\cos(x)+i\sin(x)) \\
&= (\cos((n-1)x)\cos x - \sin((n-1)x)\sin(x)) \\
&\quad + i(\cos(n-1)x\sin(x) + \sin((n-1)x)\cos(x)) \\
&= \cos((n-1)x+x) + i\sin((n-1)x+x) \\
&= \cos(nx)+i\sin(nx).
\end{aligned}
$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$

with $\alpha = (n-1)x$ and $\beta = x$.

Thus $P(n)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for all $n \geq 1$, i.e.

$$(\cos(x)+i\sin(x))^n = (\cos(nx)+i\sin(nx)).$$

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