Extracts from MATH 142 Final Exam (Spring, 1995)

On this part of the exam no calculators or computers may be used. For full credit you must show enough work so that your methods and reasoning are clear.

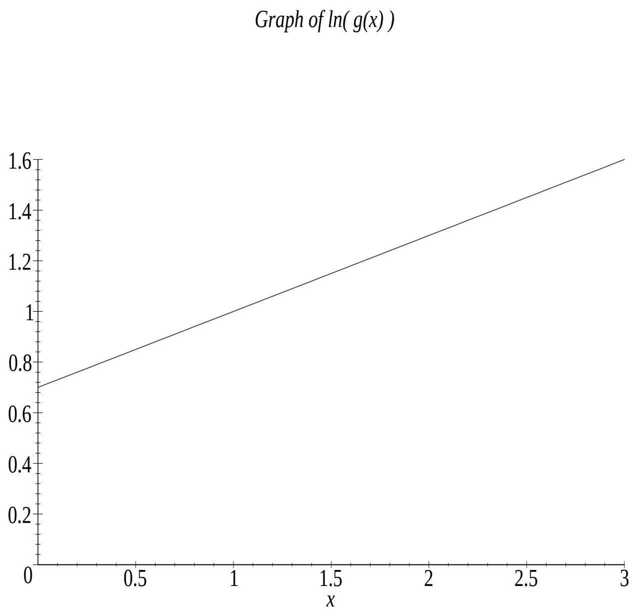
1. Compute the derivative.
   a. $\frac{d}{dr}\sin(\pi r)e^{-r}$
   b. $\frac{d}{dt}\ln\sqrt[3]{te^t}$

2. Compute each integral.
   a. $\int\left(\cos(\frac{x}{4})+\frac{6}{x^3}\right)\,dx$
   b. $\int\tan 3z\,dz$
   c. $\int x^2\ln x\,dx$
   d. $\int_0^\infty r^2e^{-r^3}\,dr$
   e. $\int\frac{t+14}{t^2+3t-4}\,dt$

3. Compute the average value of the function $f(w)=\dfrac{1}{16+w^2}$ on the interval $0\le w\le 4$.

On the remainder of the exam calculators or Maple may be used. For full credit you must show enough work so that your methods and reasoning are clear. If you use Maple or a calculator, say what exactly you are having the machine do for you.

4. Suppose the graph of $\ln(g(x))$ is the one shown below. Find the formula for $g(x)$ itself, and sketch its graph.



5. Compute the area enclosed between the curves $y=2x^3$ and $y=-x^2+4x+3$.

6. Sketch a graph that is consistent with the **all** the following information:
   - $g(x)$ is defined at all x except for $x = 0$
   - $g(-2) = -4$
   - $g'(x) < 0$ for $x < -2$ and for $x > 0$
   - $g'(-2) = 0$
   - $g'(x) > 0$ for $-2 < x < 0$
   - $g''(x) < 0$ for $x < -3$
   - $g''(x) > 0$ for $-3 < x < 0$ and for $x > 0$

7. A particle moves so that $\mathbf{r}(t) = 3\cos t\hat{\imath} + \sin t\hat{\jmath} = (3\cos t, \sin t)$.
   a. Compute the velocity vector $\mathbf{v}(t)$ and the speed.
   b. Give an equation in the variables x and y that describes the path of the motion. Sketch the path for $0 \le t \le \pi$, and indicate the direction of motion.

8. We find a Riemann sum approximation $\int_0^{\pi/6} \sec^3 x\, dx \approx 0.60827$ by using the right endpoints of 500 equal size subintervals. Does this Riemann sum over- or under-estimate the integral?

9. The local gas station/video rental/convenience store has three staff persons working from opening at 7 am to 2 pm, two working from 2 pm to 4 pm, four working from 4 pm to 10 pm, and one from 10 pm to closing at 1 am. One fine day everyone works as usual, except the two on duty from 2 pm to 4 pm lock up at 3 pm and sneak out to play street hockey on the roof for an hour. Graph the work done (in staff-hours) over the course of this day as a function of time. What is the average number of staff on duty that particular day?

10. a. Compute the Taylor polynomial of degree 2 for $\sqrt{1+x}$, centered at $x = 0$.
    b. The electric potential V, at a distance r along the axis perpendicular to the center of a charged disc of radius a and charge density σ, is $V = 2\pi\sigma(\sqrt{r^2 + a^2} - r)$. If r is large in comparison to a, then a/r is close to 0. Use part a. to give an approximation for $\sqrt{r^2 + a^2} - r$ as r times a polynomial in powers of a/r.

Unconstrained population growth

In many cases, the rate of change of a variable quantity is proportional to the quantity itself. Consider human population. If a city of 100,000 is increasing at a rate of 1,500 persons/year, we would expect a similar city of 200,000 to be increasing at a rate of 3,000 persons/year and a city of 600,000 to be increasing at a rate of what?

1. For the cities described above, if P represents the population at time t, and the **net growth rate** P' is proportional to P, compute the value of the proportionality constant, which is customarily denoted by the letter k in chemistry and math, and by the letter r in biology. That is, if $P' = rP$, determine the value of r. What are the units for r? This constant is generally called the **per capita growth rate** or **intrinsic growth rate**). Why is the

terminology “per capita” appropriate? (By the way, you may be tempted to think of P' as the derivative of P–and it is–but for the moment this is not as useful as thinking of it as the net growth rate. In any event since we do not have a formula for P we can't “derive” P'.)

2. In 1985 the per capita growth rate of Poland was 9 persons/year per 1000 persons. For Afghanistan it was 21.6 persons/year per 1000 persons. If P denotes the population of Poland and A the population of Afghanistan, write the rate equations for P' and A' that govern the growth of these two populations.

3. In 1985 the population of Poland was 37.5 million, and of Afghanistan 15 million. What were the net growth rates in 1985? Is it true that a population with a larger per capita growth rate also has a larger net growth rate? What is the correct conclusion that one can draw in comparing populations with different per capita growth rates?

4. If we use Δt to denote a time interval, and ΔP to denote the change in the population of Poland during this time interval, write an equation that shows how the quantities ΔP, P', and Δt are connected to one another (hint: if you consider the units for these quantities, that may help). Use this equation to compute the population of Poland in 1990, based on the available information.

5. Critique the solution you have just given. Can you see a way to to get a better estimate for the population in 1990? (Hint: what happens to P' as soon as P changes?) Use your improved method, say with 10 steps, to track P from 1985 to 1990. Do the same for A. Sketch the graphs illustrating the population growth. The method is credited to Euler, and the graphs that you obtained are called **piecewise linear**.

The analysis that you made above can be put into a more familiar setting. Recall that $P' = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t}$. So if Δt is small, P' is approximately equal to $\frac{\Delta P}{\Delta t}$ or, equivalently, and much more usefully, $\boxed{\Delta P \approx P' \Delta t}$. What you have done above is apply this approximation over and over again: an initial value of P gives an initial value of $P' = rP$, which gives ΔP, which enables us to update the value of P, which gives a new P', and on and on it goes.

6. Find the analytic solution (formula) for this differential equation $P' = rP$, that is, find a formula for P as a function of t. (Hint: rewrite $\frac{dP}{dt} = rP$, separate variables, integrate both sides to “undo” the derivative, consolidate constants, do some algebra. The constant of integration can be expressed in terms of P_0, the value of P at $t = 0$.)

7. Apply your formula to the cases of Poland and Afghanistan. What does the formula predict for the population of each in 1990? How long will it take for each population to double in size? What is plainly wrong with this model of population growth?