> restart;

Homework Solutions for Section 5.3 -- Forced Vibrations

Math 242
Spring 1995

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REMARKS:

Note that each problem contains some textual explanation. Think about how worthless this would be if there were no words, or if the plots were not labelled. Now, think about how difficult it is for me to read some of your homework solutions.

EVERY SOLUTION MUST CONTAIN SOME WRITTEN DESCRIPTION

#1 (p. 216)

As usual, we begin by finding the ODE that models the situation that is described. After solving the equation we will then be able to identify the steady-state solution.

```
> ODE := 8*diff( x(t), t,t ) + 3*diff( x(t), t ) + 40*x(t) = cos(2*t);
```

$$ODE := 8\left(\frac{\partial^2}{\partial t^2}\mathrm{x}(t)\right) + 3\left(\frac{\partial}{\partial t}\mathrm{x}(t)\right) + 40\,\mathrm{x}(t) = \cos(2\,t)$$

While we did not discuss this fact in class, it is a fact that the steady-state solution does not depend on the initial conditions. Thus, let's simply find the general solution to the ODE.

```
> EOM := dsolve( ODE, x(t) );
```

Just as we saw with the example in Friday's class, this solution is not in a form that we accept. We can get around this by manually constructing the general solution of the homogeneous equation and a particular solution to the nonhomogeneous equation.

First, the general solution to the homogeneous equation

```
> hSOLN := rhs( dsolve( lhs(ODE)=0, x(t) ) );
```

$$hSOLN := \_C1\, e^{(-3/16\,t)}\cos\left(\frac{1}{16}\sqrt{1271}\,t\right) + \_C2\, e^{(-3/16\,t)}\sin\left(\frac{1}{16}\sqrt{1271}\,t\right)$$

Next, use the method of undetermined coefficients to find a particular solution

```
> guess := A*cos(2*t) + B*sin(2*t);
```

$$guess := A\cos(2\,t) + B\sin(2\,t)$$

```
> GOAL := simplify( subs( x(t)=guess, ODE ) );
```

$$GOAL := 8\,A\cos(2\,t) + 8\,B\sin(2\,t) - 6\,A\sin(2\,t) + 6\,B\cos(2\,t) = \cos(2\,t)$$

```
> EQS := { 8*A + 6*B = 1, 8*B - 6*A = 0 };
```

$$EQS := \{8\,B - 6\,A = 0, 8\,A + 6\,B = 1\}$$

```
> COEFFS := solve( EQS, {A,B} );
```

$$COEFFS := \left\{B = \frac{3}{50}, A = \frac{2}{25}\right\}$$

```
> pSOLN := subs( COEFFS, guess );
```

$$pSOLN := \frac{2}{25}\cos(2\,t) + \frac{3}{50}\sin(2\,t)$$

Thus, the general solution is

```
> gSOLN := hSOLN + pSOLN;
```

$$gSOLN := \_C1\,e^{(-3/16\,t)}\cos\left(\frac{1}{16}\sqrt{1271}\,t\right) + \_C2\,e^{(-3/16\,t)}\sin\left(\frac{1}{16}\sqrt{1271}\,t\right) + \frac{2}{25}\cos(2\,t) + \frac{3}{50}\sin(2\,t)$$

The steady-state solution is that part of this expression that does not decay to zero as t increases to infinity. That is, the steady-state solution is:

```
> SS := pSOLN;
```

$$SS := \frac{2}{25}\cos(2\,t) + \frac{3}{50}\sin(2\,t)$$

4 (p. 216)

This problem comes in two parts. For the first part we need to find the equation of motion, i.e. solve an IVP. The second part asks for the resonance frequency of the system. This will take a little more work.

```
> restart;
> ODE := 2*diff( x(t), t,t ) + 5*diff( x(t), t ) + 98*x(t) = 3/10*cos(t);
```

$$ODE := 2\left(\frac{\partial^2}{\partial t^2}\mathrm{x}(t)\right) + 5\left(\frac{\partial}{\partial t}\mathrm{x}(t)\right) + 98\,\mathrm{x}(t) = \frac{3}{10}\cos(t)$$

```
> IC1 := x(0) = 5;
```

$$IC1 := \mathrm{x}(0) = 5$$

```
> IC2 := D(x)(0) = 0;
```

$$IC2 := \mathrm{D}(x)(0) = 0$$

By now I won't even attempt to use dsolve to directly find this solution.

```
> hSOLN := rhs( dsolve( lhs(ODE)=0, x(t) ) );
```

$$hSOLN := \_C1\,\mathrm{e}^{(-5/4\,t)}\cos\left(\frac{1}{4}\sqrt{759}\,t\right) + \_C2\,\mathrm{e}^{(-5/4\,t)}\sin\left(\frac{1}{4}\sqrt{759}\,t\right)$$

```
> guess := A*cos(t) + B*sin(t);
```

$$guess := A\cos(t) + B\sin(t)$$

```
> GOAL := simplify( subs( x(t) = guess, ODE ) );
```

$$GOAL := 96\,A\cos(t) + 96\,B\sin(t) - 5\,A\sin(t) + 5\,B\cos(t) = \frac{3}{10}\cos(t)$$

```
> EQS := { 96*A + 5*B = 3/10, 96*B - 5*A = 0 };
```

$$EQS := \left\{96\,B - 5\,A = 0,\ 96\,A + 5\,B = \frac{3}{10}\right\}$$

```
> COEFFS := solve( EQS, {A,B} );
```

$$COEFFS := \left\{B = \frac{3}{18482},\ A = \frac{144}{46205}\right\}$$

```
> pSOLN := subs( COEFFS, guess );
```

$$pSOLN := \frac{144}{46205}\cos(t) + \frac{3}{18482}\sin(t)$$

```
> gSOLN := hSOLN + pSOLN;
```

$$gSOLN := \_C1\, e^{(-5/4\,t)} \cos\left(\frac{1}{4}\sqrt{759}\ t\right) + \_C2\, e^{(-5/4\,t)} \sin\left(\frac{1}{4}\sqrt{759}\ t\right) + \frac{144}{46205}\cos(t) + \frac{3}{18482}\sin(t)$$

And, finally, use the initial conditions to solve for _C1 and _C2.

```
> EQS := simplify( { subs( t=0, gSOLN )=5, subs( t=0, diff(gSOLN,t) ) = 0 } );
```

$$EQS := \left\{ \_C1 + \frac{144}{46205} = 5, -\frac{5}{4}\_C1 + \frac{1}{4}\_C2\sqrt{759} + \frac{3}{18482} = 0 \right\}$$

```
> CONSTS := solve( EQS, {_C1,_C2} );
```

$$CONSTS := \left\{ \_C2 = \frac{230875}{7013919}\sqrt{759}, \_C1 = \frac{230881}{46205} \right\}$$

The equation of motion is

```
> EOM := subs( CONSTS, gSOLN );
```

$$EOM := \frac{230881}{46205} e^{(-5/4\,t)} \cos\left(\frac{1}{4}\sqrt{759}\ t\right) + \frac{230875}{7013919}\sqrt{759}\ e^{(-5/4\,t)} \sin\left(\frac{1}{4}\sqrt{759}\ t\right) + \frac{144}{46205}\cos(t) + \frac{3}{18482}\sin(t)$$

Before beginning the second part of this problem, note that the steady-state solution is

```
> SS := pSOLN;
```

$$SS := \frac{144}{46205}\cos(t) + \frac{3}{18482}\sin(t)$$

which can be re-written as

```
> SSa := A*sin(t+phi);
```

$$SSa := A\sin(t + \phi)$$

where

```
> A := sqrt( (144/46205)^2 + (3/18482)^2 );
```

$$A := \frac{3}{92410}\sqrt{9241}$$

```
> phi := arctan( (144/46205), (3/18482) );
  evalf( " );
```

$$\phi := \arctan\left(\frac{96}{5}\right)$$

$$1.518760012$$

The frequency gain for this solution is

```
> freq_gain := (3*sqrt(9241)/92410) / (3/10);
  evalf( freq_gain );
```

$$freq\_gain := \frac{1}{9241}\sqrt{9241}$$

$$.01040256684$$

To find the resonance frequency requires finding the frequency for which the frequency gain is maximized. In order to find the steady-state solution we repeat the above process with forcing term: 3/10*cos(gamma*t). But, before you just start from scratch, note that the general solution to the homogeneous equation is the same. All we need to do is re-determine a particular solution.

```
> constants := op( {constants} minus {gamma} ):
  A := 'A':
  B := 'B':
  phi := 'phi':
> ODE := 2*diff( x(t), t,t ) + 5*diff( x(t), t ) + 98*x(t) = 3/10*cos(gamma*t);
```

$$ODE := 2\left(\frac{\partial^2}{\partial t^2}\mathrm{x}(t)\right) + 5\left(\frac{\partial}{\partial t}\mathrm{x}(t)\right) + 98\,\mathrm{x}(t) = \frac{3}{10}\cos(\gamma\,t)$$

```
> hSOLN;
```

$$\_C1\,e^{(-5/4\,t)}\cos\left(\frac{1}{4}\sqrt{759}\,t\right) + \_C2\,e^{(-5/4\,t)}\sin\left(\frac{1}{4}\sqrt{759}\,t\right)$$

```
> guess := A*cos(gamma*t) + B*sin(gamma*t);
```

$$guess := A\cos(\gamma\,t) + B\sin(\gamma\,t)$$

```
> GOAL := simplify( subs( x(t)=guess, ODE ) );
```

$$GOAL := -2\,A\cos(\gamma\,t)\,\gamma^2 - 2\,B\sin(\gamma\,t)\,\gamma^2 - 5\,A\sin(\gamma\,t)\,\gamma + 5\,B\cos(\gamma\,t)\,\gamma + 98\,A\cos(\gamma\,t) + 98\,B\sin(\gamma\,t) = \frac{3}{10}\cos(\gamma\,t)$$

```
> EQS := { -2*A*gamma^2 + 5*B*gamma + 98*A = 3/10,
           -2*B*gamma^2 - 5*A*gamma + 98*B = 0 };
```

$$EQS := \left\{-2\,B\,\gamma^2 - 5\,A\,\gamma + 98\,B = 0,\ -2\,A\,\gamma^2 + 5\,B\,\gamma + 98\,A = \frac{3}{10}\right\}$$

```
> COEFFS := solve( EQS, {A,B} );
```

$$COEFFS := \left\{B = \frac{3}{2}\,\frac{\gamma}{-367\,\gamma^2 + 4\,\gamma^4 + 9604},\ A = -\frac{3}{5}\,\frac{\gamma^2 - 49}{-367\,\gamma^2 + 4\,\gamma^4 + 9604}\right\}$$

> **pSOLN := subs(COEFFS, guess);**

$$pSOLN := -\frac{3}{5}\frac{(\gamma^2-49)\cos(\gamma t)}{-367\gamma^2+4\gamma^4+9604}+\frac{3}{2}\frac{\gamma\sin(\gamma t)}{-367\gamma^2+4\gamma^4+9604}$$

Thus, the general solution -- the equation of motion -- is

> **gSOLN := hSOLN + pSOLN;**

$$gSOLN := \_C1\, e^{(-5/4\, t)}\cos\left(\frac{1}{4}\sqrt{759}\, t\right)+\_C2\, e^{(-5/4\, t)}\sin\left(\frac{1}{4}\sqrt{759}\, t\right)-\frac{3}{5}\frac{(\gamma^2-49)\cos(\gamma t)}{-367\gamma^2+4\gamma^4+9604}$$
$$+\frac{3}{2}\frac{\gamma\sin(\gamma t)}{-367\gamma^2+4\gamma^4+9604}$$

As usual, the steady-state solution is the particular solution. However, to be most useful note that the steady-state solution can be written as

> **SS := A*sin(gamma*t + phi);**

$$SS := A\sin(\gamma t+\phi)$$

with

> **A := sqrt((-3/5*(gamma^2-49)/(-367*gamma^2 + 4*gamma^4 + 9604))^2 + (3/2*gamma/(-367*gamma^2 + 4*gamma^4 + 9604))^2);**

$$A := \frac{3}{10}\sqrt{4\frac{(\gamma^2-49)^2}{(-367\gamma^2+4\gamma^4+9604)^2}+25\frac{\gamma^2}{(-367\gamma^2+4\gamma^4+9604)^2}}$$

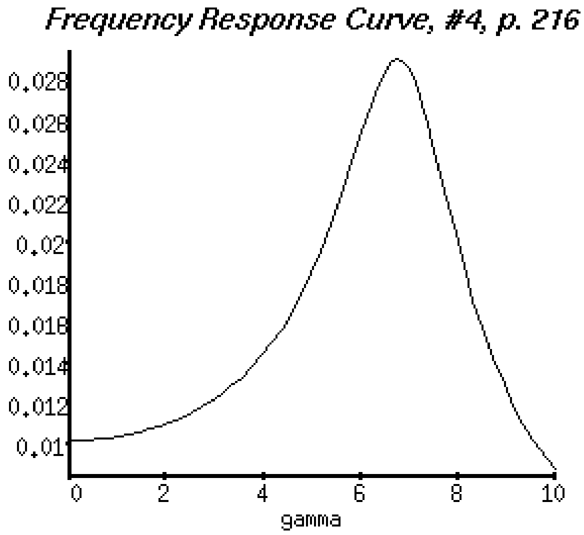
> **# phi := Wait! I don’t need phi to find the frequency gain!**

The frequency gain for a general forcing term is

```
> M := simplify( A / (3/10) );
```

$$M := \sqrt{\frac{1}{-367\,\gamma^2 + 4\,\gamma^4 + 9604}}$$

```
> plot( M,  gamma = 0..10, title=‘Frequency Response Curve, #4, p. 216‘ );
```



This plot shows that the resonance frequency occurs for near gamma=6.75. Calculus will provide an exact answer

```
> diff( M, gamma );
```

$$-\frac{1}{2}\,\frac{-734\,\gamma + 16\,\gamma^3}{\sqrt{\frac{1}{-367\,\gamma^2 + 4\,\gamma^4 + 9604}}\,\left(-367\,\gamma^2 + 4\,\gamma^4 + 9604\right)^2}$$

```
> solve( diff( M, gamma )=0, gamma );
```

$$0,\ \mathrm{RootOf}(-367 + 8\,\_Z^2)$$

```
> allvalues( ["] );
```

$$\left[0, \frac{1}{4}\sqrt{734}\right], \left[0, -\frac{1}{4}\sqrt{734}\right]$$

Thus, the exact location of resonance is gamma=sqrt(734)/4. That is, the resonance frequency is sqrt(734)/(8*Pi).

```
>
```

6. (p. 216)

To find the frequency resonse curve, we have to solve the following ODE with a forcing term of cos(gamma*t).

```
> restart;
  constants := op( {constants} minus {gamma} ):
> ODE := 2*diff( x(t), t,t ) +3*diff( x(t), t ) + 3*x(t) = cos(gamma*t);
```

$$ODE := 2\left(\frac{\partial^2}{\partial t^2}\mathrm{x}(t)\right) + 3\left(\frac{\partial}{\partial t}\mathrm{x}(t)\right) + 3\,\mathrm{x}(t) = \cos(\gamma\, t)$$

Here’s how the solution can be found

```
> hSOLN := rhs( dsolve( lhs(ODE)=0, x(t) ) );
```

$$hSOLN := \_C1\, e^{(-3/4\, t)} \cos\left(\frac{1}{4}\sqrt{15}\, t\right) + \_C2\, e^{(-3/4\, t)} \sin\left(\frac{1}{4}\sqrt{15}\, t\right)$$

```
> guess := A*cos(gamma*t) + B*sin(gamma*t);
```

$$guess := A\cos(\gamma\, t) + B\sin(\gamma\, t)$$

```
> GOAL := simplify( subs( x(t)=guess, ODE ) );
```

$GOAL :=$

$$-2\,A\cos(\gamma\, t)\,\gamma^2 - 2\,B\sin(\gamma\, t)\,\gamma^2 - 3\,A\sin(\gamma\, t)\,\gamma + 3\,B\cos(\gamma\, t)\,\gamma + 3\,A\cos(\gamma\, t) + 3\,B\sin(\gamma\, t) = \cos(\gamma\, t)$$

```
> EQS := { -2*A*gamma^2 + 3*B*gamma + 3*A = 1, -2*B*gamma^2 - 3*A*gamma + 3*B = 0 };
```

$$EQS := \left\{-2\,B\,\gamma^2 - 3\,A\,\gamma + 3\,B = 0,\ -2\,A\,\gamma^2 + 3\,B\,\gamma + 3\,A = 1\right\}$$

```
> COEFFS := solve( EQS, {A,B} );
```

$$COEFFS := \left\{A = -\frac{2\,\gamma^2 - 3}{-3\,\gamma^2 + 4\,\gamma^4 + 9},\ B = 3\,\frac{\gamma}{-3\,\gamma^2 + 4\,\gamma^4 + 9}\right\}$$

```
> pSOLN := subs( COEFFS, guess );
```

$$pSOLN := -\frac{\left(2\,\gamma^2 - 3\right)\cos(\gamma\, t)}{-3\,\gamma^2 + 4\,\gamma^4 + 9} + 3\,\frac{\gamma\sin(\gamma\, t)}{-3\,\gamma^2 + 4\,\gamma^4 + 9}$$

The amplitude of the particular solution is

```
> A := sqrt( ( -(2*gamma^2-3)/(-3*gamma^2+4*gamma^4+9) )^2 + ( 3*gamma/(-3*gamma^2+4*
  gamma^4+9) )^2 );
```

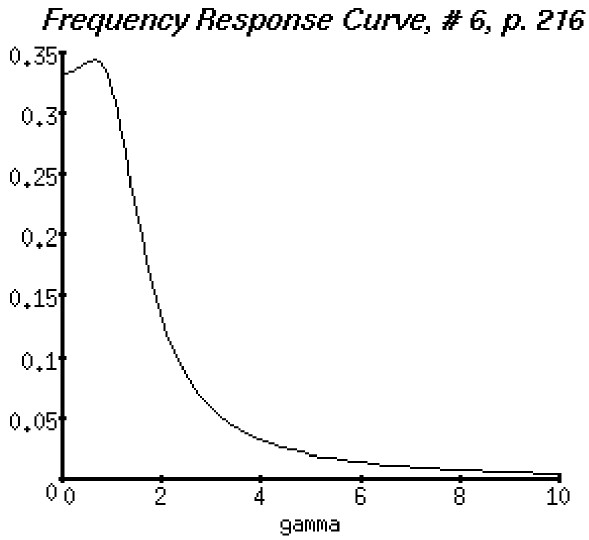
$$A := \sqrt{\frac{(2\gamma^2 - 3)^2}{(-3\gamma^2 + 4\gamma^4 + 9)^2} + 9\frac{\gamma^2}{(-3\gamma^2 + 4\gamma^4 + 9)^2}}$$

Since the coefficient of the forcing term is 1, the frequency gain is just the amplitude of the steady-state solution

```
> M := simplify( A );
```

$$M := \sqrt{\frac{1}{-3\gamma^2 + 4\gamma^4 + 9}}$$

```
> plot( M, gamma=0..10, title=‘Frequency Response Curve, # 6, p. 216‘ );
```




```
>
```

7 (p. 216)

Believe it or not, this is the easiest of the problems from this section.

```
> restart;
> ODE := diff( x(t), t,t )  + 9*x(t) = 2*cos(3*t);
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}\, \mathrm{x}(t)\right) + 9\, \mathrm{x}(t) = 2\cos(3\, t)$$

```
> IC1 := x(0) = 1;
```

$$IC1 := \mathrm{x}(0) = 1$$

```
> IC2 := D(x)(0) = 0;
```

$$IC2 := \mathrm{D}(x)(0) = 0$$

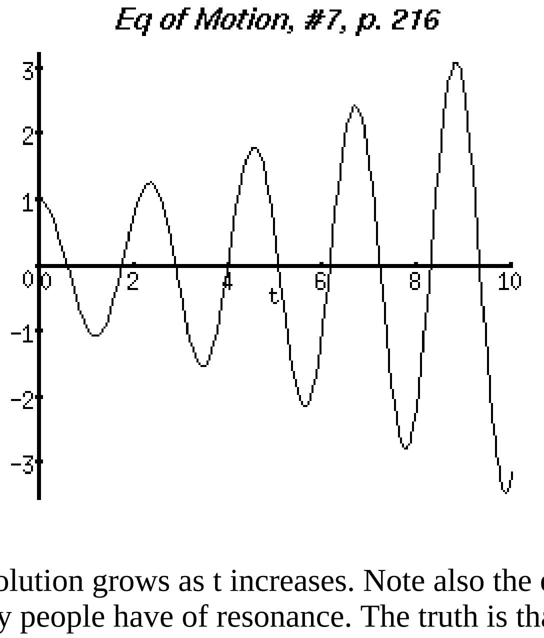
The equation of motion is obtained as the solution to this IVP.

```
> EOM := rhs( dsolve( { ODE, IC1, IC2 }, x(t) ) );
```

$$EOM := \cos(3\, t) + \frac{1}{3}\sin(3\, t)\, t$$

Before finding the exact time at which the mass has its maximum displacement from equilibrium (either positive or negative), let’s make a sketch of the equation of motion

```
> plot( EOM, t=0..10, title=‘Eq of Motion, #7, p. 216‘ );
```



Note that the magnitude of the solution grows as t increases. Note also the extra t in the particular solution. This behavior is the picture many people have of resonance. The truth is that resonace causes such growing solutions only in the undamped case.

END OF WORKSHEET

```
>
```