```
> restart;
```

Homework Solutions for Section 5.2 -- Dampled Free Vibrations

Math 242
Spring 1995

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REMARKS:

Note that each problem contains some textual explanation. Think about how worthless this would be if there were no words, or if the plots were not labelled. Now, think about how difficult it is for me to read some of your homework solutions.

EVERY SOLUTION MUST CONTAIN SOME WRITTEN DESCRIPTION

#2 (p. 210)

The first two problems request the solution of three IVP and a display of their graphs. To simplify matters, let's enter the general initial value problem

```
> ODE := diff( x(t), t,t ) + b*diff( x(t), t ) + 64*x(t) = 0;
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}\mathrm{x}(t)\right) + b\left(\frac{\partial}{\partial t}\mathrm{x}(t)\right) + 64\,\mathrm{x}(t) = 0$$

Similar conversion of the initial displacement to meters leads to the initial conditions

```
> IC1 := x(0) = 1;
```

$$IC1 := \mathrm{x}(0) = 1$$

```
> IC2 := D(x)(0) = 0;
```

$$IC2 := \mathrm{D}(x)(0) = 0$$

```
>
```

The problem asks for several characteristics of the equation of motion (i.e. the solution). I suggest starting from a sketch of the solution.

```
> EOMa := rhs( dsolve( { subs(b=10, ODE), IC1, IC2 }, x(t) ) );
```

$$EOMa := e^{(-5\,t)}\cos\left(\sqrt{39}\,t\right) + \frac{5}{39}\sqrt{39}\,e^{(-5\,t)}\sin\left(\sqrt{39}\,t\right)$$

```
> EOMb := rhs( dsolve( { subs(b=16, ODE), IC1, IC2 }, x(t) ) );
```

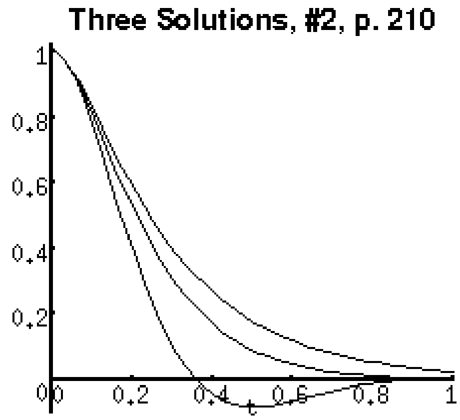
$$EOMb := e^{(-8\,t)} + 8\,e^{(-8\,t)}\,t$$

```
> EOMc := rhs( dsolve( { subs(b=20, ODE), IC1, IC2 }, x(t) ) );
```

$$EOMc := \frac{4}{3}e^{(-4\,t)} - \frac{1}{3}e^{(-16\,t)}$$

From these three solutions it is apparent that the first (b=6) is underdamped, the second (b=16) is critically damped, and the third solution (b=20) is overdamped. Here are the sketches all on the same axes.

```
> plot( { EOMa, EOMb, EOMc }, t=0..1, title=‘Three Solutions, #2, p. 210‘ );
```




```
>
```

NOTE: You would NOT have obtained correct answers if you had solved the IVP with arbitrary constant, b, and then substituted the different values of b into the solution.

#3 (p. 210)

This question is the same as #2, except there are different values given for the spring constant, k.

```
> restart;
> ODE := diff( x(t), t,t ) + 10*diff( x(t), t ) + k*x(t) = 0;
```

$$ODE := \left(\frac{\partial^2}{\partial t^2} x(t)\right) + 10\left(\frac{\partial}{\partial t} x(t)\right) + k\, x(t) = 0$$

```
> IC1 := x(0) = 1;
```

$$IC1 := x(0) = 1$$

```
> IC2 := D(x)(0) = 0;
```

$$IC2 := D(x)(0) = 0$$

```
> EOMa := rhs( dsolve( { subs(k=20, ODE), IC1, IC2 }, x(t) ) );
```

$$EOMa := \left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right) e^{\left(\left(-5 + \sqrt{5}\right) t\right)} + \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right) e^{\left(-\left(5 + \sqrt{5}\right) t\right)}$$

```
> EOMb := rhs( dsolve( { subs(k=25, ODE), IC1, IC2 }, x(t) ) );
```

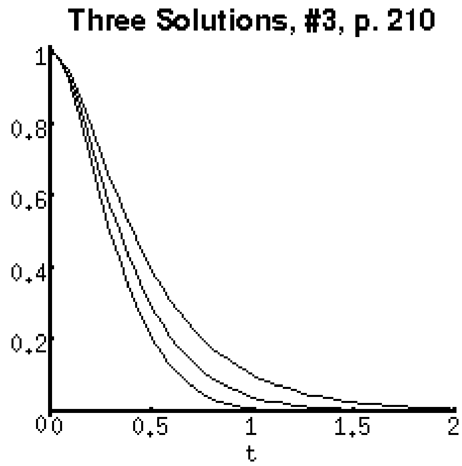
$$EOMb := e^{(-5\,t)} + 5\, e^{(-5\,t)}\, t$$

```
> EOMc := rhs( dsolve( { subs(k=30, ODE), IC1, IC2 }, x(t) ) );
```

$$EOMc := e^{(-5\,t)} \cos\left(\sqrt{5}\, t\right) + \sqrt{5}\, e^{(-5\,t)} \sin\left(\sqrt{5}\, t\right)$$

Note that now the largest value of k, k=30, is an underdamped system, the middle value of k, k=25, is critically damped and the smallest value of k, k=20, leads to an overdamped solution. Here is a plot of all three solutions (note that you can‘t see the oscillations in EOMc).

```
> plot( { EOMa, EOMb, EOMc }, t=0..2, title=‘Three Solutions, #3, p. 210‘ );
```




```
>
```

6. (p. 210)

This situation is modelled by the IVP

> restart;

> ODE := 20*diff(x(t), t,t) +140*diff(x(t), t) + 200*x(t) = 0;

$$ODE := 20\left(\frac{\partial^2}{\partial t^2}\,\mathrm{x}(t)\right) + 140\left(\frac{\partial}{\partial t}\,\mathrm{x}(t)\right) + 200\,\mathrm{x}(t) = 0$$

> IC1 := x(0) = 1/4;

$$IC1 := \mathrm{x}(0) = \frac{1}{4}$$

> IC2 := D(x)(0) = -1;

$$IC2 := \mathrm{D}(x)(0) = -1$$

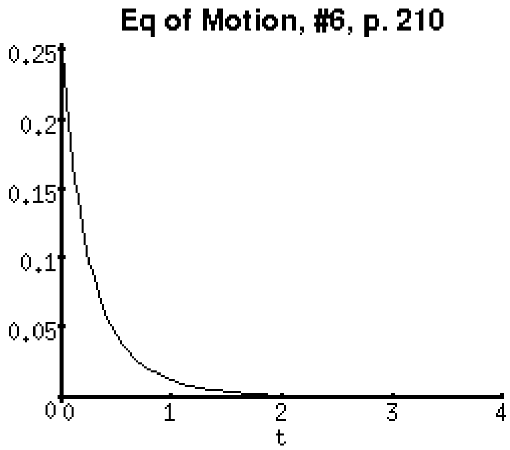
Solving this IVP leads to the equation of motion

> EOM := rhs(dsolve({ ODE, IC1, IC2 }, x(t)));

$$EOM := \frac{1}{6}\,\mathrm{e}^{(-5\,t)} + \frac{1}{12}\,\mathrm{e}^{(-2\,t)}$$

We seek the time at which the mass returns to its equilibrium solution (i.e. x(t) = 0). Let's get an estimate from the graph

> plot(EOM, t=0..4, title='Eq of Motion, #6, p. 210');



From this plot it appears the mass NEVER returns to its equilbrium position. Now, the mass gets close to the equilibrium position, but never does it exactly reach the same point.

That this is the case can also be seen from direct examination of the solution. Both terms are positive, and are added
together. There is no way to add two positive numbers to reach a value of 0.

>

7 (p. 210)

First, the ODE and initial conditions

```
> restart;
> ODE := 2*diff( x(t), t,t ) + 8*sqrt(5)*diff( x(t), t ) + 40*x(t) = 0;
```

$$ODE := 2\left(\frac{\partial^2}{\partial t^2}\,\mathrm{x}(t)\right) + 8\sqrt{5}\left(\frac{\partial}{\partial t}\,\mathrm{x}(t)\right) + 40\,\mathrm{x}(t) = 0$$

```
> IC1 := x(0) = 1/10;
```

$$IC1 := \mathrm{x}(0) = \frac{1}{10}$$

```
> IC2 := D(x)(0) = 2;
```

$$IC2 := \mathrm{D}(x)(0) = 2$$

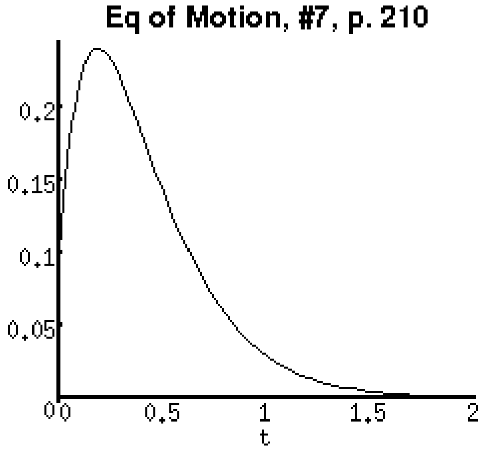
The equation of motion is obtained as the solution to this IVP.

```
> EOM := rhs( dsolve( { ODE, IC1, IC2 }, x(t) ) );
```

$$EOM := \frac{1}{10}\,e^{(-2\sqrt{5}\,t)} + \left(\frac{1}{5}\sqrt{5} + 2\right)e^{(-2\sqrt{5}\,t)}\,t$$

Before finding the exact time at which the mass has its maximum displacement from equilibrium (either positive or negative), let’s make a sketch of the equation of motion

```
> plot( EOM, t=0..2, title=‘Eq of Motion, #7, p. 210‘ );
```



This shows that the maximum displacement from equilibrium occurs near t=0.185 seconds when the mass is approximately 0.24 meters below the equilibrium.

To find an exact answer, we need to find the (unique) critical point of this solution.

```
> diff( EOM, t );
```

$$-\frac{1}{5}\sqrt{5}\, e^{(-2\sqrt{5}\, t)} - 2\left(\frac{1}{5}\sqrt{5} + 2\right)\sqrt{5}\, e^{(-2\sqrt{5}\, t)}\, t + \left(\frac{1}{5}\sqrt{5} + 2\right)e^{(-2\sqrt{5}\, t)}$$

```
> CP := solve( "=0, t );
```

$$CP := -\frac{1}{19} + \frac{2}{19}\sqrt{5}$$

```
> evalf( CP );
```

$$.1827439977$$

The (approximate) maximum displacement is

```
> evalf( subs( t=CP, EOM ) );
```

$$.2416724288$$

```
>
```

END OF WORKSHEET

```
>
```