```
> restart;
```

Homework Solutions for Section 5.1 -- Mechanical Vibrations and Simple Harmonic Motion

Math 242
Spring 1995

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7 April 1995

REMARKS:

Note that each problem contains some textual explanation. Think about how worthless this would be if there were no words, or if the plots were not labelled. Now, think about how difficult it is for me to read some of your homework solutions.

EVERY SOLUTION MUST CONTAIN SOME WRITTEN DESCRIPTION

1 (p. 203)

The first step is to find the IVP that models this situation. The problem explicitly tells us that m=100kg; the other information, after converting 20cm to 0.2m, permits the determination that the spring constant is k=m*g/l = 20*9.8/0.2 = 980. Thus, the ODE is

```
> ODE := 100*diff( x(t), t,t ) + 980*x(t) = 0;
```

$$ODE := 100 \left(\frac{\partial^2}{\partial t^2} \mathrm{x}(t) \right) + 980\, \mathrm{x}(t) = 0$$

Similar conversion of the initial displacement to meters leads to the initial conditions

```
> IC1 := x(0) = 0.05;
```

$$IC1 := \mathrm{x}(0) = .05$$

```
> IC2 := D(x)(0) = 0;
```

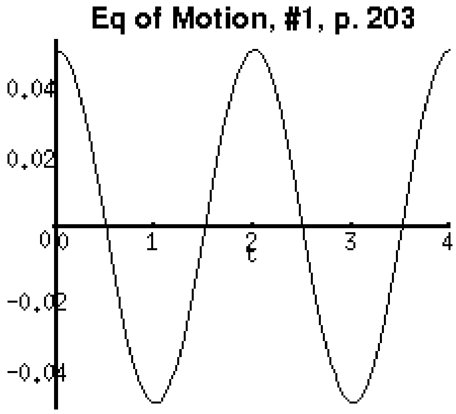
$$IC2 := \mathrm{D}(x)(0) = 0$$

The problem asks for several characteristics of the equation of motion (i.e. the solution). I suggest starting from a sketch of the solution.

```
> EOM := dsolve( { ODE, IC1, IC2 }, x(t) );
```

$$EOM := \mathrm{x}(t) = .05000000000 \cos\left(\frac{7}{5} \sqrt{5}\; t \right)$$

```
> plot( rhs( EOM ), t=0..4, title=‘Eq of Motion, #1, p. 203‘ );
```



This solution is simple enough that the amplitude, period, and natural frequency can all be read directly from the solution. The values are:

```
> amplitude :=  0.05;
```

$$amplitude := .05$$

```
> period :=  evalf( 2*sqrt(5)*Pi/7 );
```

$$period := 2.007089924$$

```
> nat_freq := 1/period;
```

$$nat\_freq := .4982337802$$

Note that there is almost as much text as math. Omit the words and you have to read my mind to determine what is being done, and why it is being done. This would not be good communication!

4. (p. 203)

This problem does NOT require the use of differential equations.

First, the period of an undamped motion is given by

```
> P := 2*Pi/omega;
```

$$P := 2\,\frac{\pi}{\omega}$$

where

```
> omega := sqrt( k/m );
```

$$\omega := \sqrt{\frac{k}{m}}$$

That is, the period can be written as

```
> P;
```

$$2\,\frac{\pi}{\sqrt{\frac{k}{m}}}$$

But, the expression listed in the book does not involve k or m. It does involve l and g.

Recall that m, k, g, and l are all related by: mg = kl, so that k/m = g/l. Making this substitution yields

```
> subs( k/m=g/l, P );
```

$$2\,\frac{\pi}{\sqrt{\frac{g}{l}}}$$

which is equivalent to the expression P = 2*Pi*sqrt(l/g) listed in the book.

5 (p. 203)

Once the spring constant is computed: k = m*g/l = 5*9.8/0.2 = 98, the ODE and initial conditions are seen to be

```
> restart;
> ODE := 5*diff( x(t), t,t ) + 98*x(t);
```

$$ODE := 5\left(\frac{\partial^2}{\partial t^2}\mathrm{x}(t)\right) + 98\,\mathrm{x}(t)$$

```
> IC1 := x(0) = 1/10;
```

$$IC1 := \mathrm{x}(0) = \frac{1}{10}$$

```
> IC2 := D(x)(0) = -1/10;
```

$$IC2 := \mathrm{D}(x)(0) = \frac{-1}{10}$$

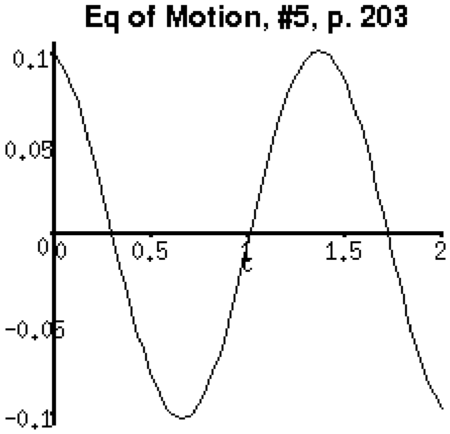
The equation of motion is obtained as the solution to this IVP.

```
> EOM := rhs( dsolve( { ODE, IC1, IC2 }, x(t) ) );
```

$$EOM := \frac{1}{10}\cos\left(\frac{7}{5}\sqrt{10}\,t\right) - \frac{1}{140}\sin\left(\frac{7}{5}\sqrt{10}\,t\right)\sqrt{10}$$

We are now in a position in which the interesting question can be answered. What does it mean for the mass to reach its minimum height? With the coordinate system chosen with positive x pointing downward, the mass reaches its minimum height when the spring is at its maximum elongation. To see where this occurs, let's look at a plot:

```
> plot( EOM, t=0..2, title='Eq of Motion, #5, p. 203' );
```



From this sketch it appears as though the maximum height is reached near t = 1.375 sec (the max elongation is 0.1029 m)

An exact answer is also possible. To see this we first write x(t) = A * sin(omega*t + phi). To achieve this we must have

```
> omega := 7*sqrt(10)/5;
  evalf( omega );
```

$$\omega := \frac{7}{5}\sqrt{10}$$

$$4.427188724$$

and A*sin(phi) = 1/10 and A*cos(phi) = -sqrt(10)/140. Solving for A and phi yields:

```
> A := sqrt( (1/10)^2 + (-sqrt(10)/140)^2 );
  evalf( A );
```

$$A := \frac{1}{140}\sqrt{206}$$

$$.1025192864$$

```
> phi := arctan( (1/10), -sqrt(10)/140 );
  evalf( phi );
```

$$\phi := -\arctan\left(\frac{7}{5}\sqrt{10}\right) + \pi$$

$$1.792945331$$

Note that I have use the "two argument arctangent" to find phi. The benefit of using the two argument arctangent function is that it will always return the angle in the correct quadrant. (Here, since sin(phi)<0 and cos(phi)>0, the angle should be in quadrant III.)

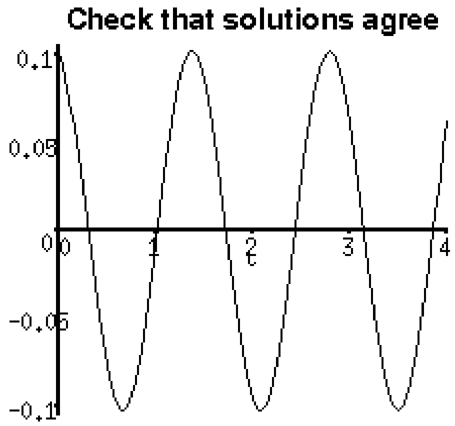
Thus, the equation of motion can equivalently be written as

```
> EOMa := A*sin( omega*t+phi );
```

$$EOMa := -\frac{1}{140}\sqrt{206}\,\sin\left(\frac{7}{5}\sqrt{10}\,t - \arctan\left(\frac{7}{5}\sqrt{10}\right)\right)$$

Let's check that the two forms of the solution agree:

> plot({ EOM, EOMa }, t=0..4, title='Check that solutions agree');



We still need to find the maximum elongation of the spring (i.e. the maximum value of x(t)). The maximum value must occur at a critical point (see the plot). Let's find a critical point of the displacement:

> vel := diff(EOMa, t);

$$vel := -\frac{1}{100}\sqrt{206}\,\cos\left(\frac{7}{5}\sqrt{10}\,t - \arctan\left(\frac{7}{5}\sqrt{10}\right)\right)\sqrt{10}$$

set this equal to 0 and solve for the time, t

> solve(vel=0, t);
evalf(");

$$-\frac{1}{14}\left(\frac{1}{2}\pi - \arctan\left(\frac{7}{5}\sqrt{10}\right)\right)\sqrt{10}$$

$$-.05017834518$$

Note that this value of t does NOT agree with our earlier estimate taken from the graph. The problem is that we've found only one (of an infinite number of) critical points of the motion. In fact, the graph suggests that this is the first time that the mass is hightest above ground. The set of all critical points of the motion can be found by finding all values of t for which the argument of cosine (in vel) is an odd multiple of Pi/2.

> CP := solve(7/5*10^(1/2)*t-arctan(7/5*10^(1/2)) = n*Pi/2, t);

$$CP := -\frac{1}{14}\left(-\arctan\left(\frac{7}{5}\sqrt{10}\right) - \frac{1}{2}n\,\pi\right)\sqrt{10}$$

> subs(n=1, evalf(CP));

$$.6594351024$$

> subs(n=3, evalf(CP));

$$1.369048550$$

Note that this critical point is very near the previous estimate for this time.

6 (p. 203)

This problem is very similar to #5. The primary difference is that we need to find the time when the mass reaches its highest point above ground. That is, when the spring has its most negative displacement from equilibrium.

We start with the ODE and initial conditions

```
> restart;
> ODE := 5*diff( x(t), t,t ) + 49/2*x(t) = 0;
```

$$ODE := 5\left(\frac{\partial^2}{\partial t^2}\,\mathrm{x}(t)\right) + \frac{49}{2}\,\mathrm{x}(t) = 0$$

```
> IC1 := x(0) = -1;
```

$$IC1 := \mathrm{x}(0) = -1$$

```
> IC2 := D(x)(0) = -1/3;
```

$$IC2 := \mathrm{D}(x)(0) = \frac{-1}{3}$$

```
> EOM := rhs( dsolve( { ODE, IC1, IC2 }, x(t) ) );
```

$$EOM := -\cos\left(\frac{7}{10}\sqrt{10}\;t\right) - \frac{1}{21}\sin\left(\frac{7}{10}\sqrt{10}\;t\right)\sqrt{10}$$

or, in the form x(t) = A*sin(omega*t + phi) with

```
> omega := 7/sqrt(10);
  evalf( omega );
```

$$\omega := \frac{7}{10}\sqrt{10}$$

$$2.213594362$$

```
> A := sqrt( (-1)^2 + (-sqrt(10)/21)^2 );
  evalf( A );
```

$$A := \frac{1}{21}\sqrt{451}$$

$$1.011274313$$

```
> phi := arctan( -1, -sqrt(10)/21 );
  evalf( " );
```

$$\phi := \arctan\left(\frac{21}{10}\sqrt{10}\right) - \pi$$

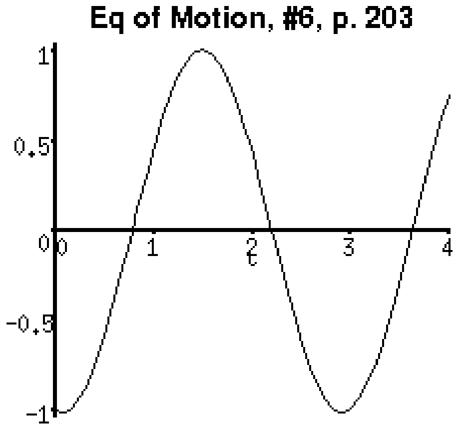
$$-1.720258011$$

(Note that phi is in the third quadrant because sin(phi) and cos(phi) are both negative.

> EOMa := A*sin(omega*t+phi);

$$EOMa := -\frac{1}{21}\sqrt{451}\,\sin\left(\frac{7}{10}\sqrt{10}\,t + \arctan\left(\frac{21}{10}\sqrt{10}\right)\right)$$

> plot(EOMa, t=0..4, title=‘Eq of Motion, #6, p. 203‘);



The first time the spring reaches its maximum height above ground is very soon after release.

> vel := diff(EOMa, t);

$$vel := -\frac{1}{30}\sqrt{451}\,\cos\left(\frac{7}{10}\sqrt{10}\,t + \arctan\left(\frac{21}{10}\sqrt{10}\right)\right)\sqrt{10}$$

> CP := solve(7/10*10^(1/2)*t+arctan(21/10*10^(1/2)) = n*Pi/2, t);

$$CP := -\frac{1}{7}\left(\arctan\left(\frac{21}{10}\sqrt{10}\right) - \frac{1}{2}\,n\,\pi\right)\sqrt{10}$$

> evalf(subs(n=1, CP));

$$.06751990635$$

> evalf(subs(n=3, CP));

$$1.486746802$$

This first of these is the correct time.

The mass first reaches its maximum height above ground only 0.06752 seconds after the mass is released.

7 (p. 203)

The formulation and solution of the differential equation is the same as the previous two problems. All that is differeent is the information to be extracted from the equation of motion. In this problem we seek the time the mass first returns to the equilibrium position (i.e. x = 0).

Let’s formulate and solve.

```
> restart;
> ODE := 1/2*diff( x(t), t,t ) + 49/20*x(t) = 0;
```

$$ODE := \frac{1}{2}\left(\frac{\partial^2}{\partial t^2}\,\mathrm{x}(t)\right) + \frac{49}{20}\,\mathrm{x}(t) = 0$$

```
> IC1 := x(0) = 1/2;
```

$$IC1 := \mathrm{x}(0) = \frac{1}{2}$$

```
> IC2 := D(x)(0) = -1/2;
```

$$IC2 := \mathrm{D}(x)(0) = \frac{-1}{2}$$

```
> EOM := rhs( dsolve( { ODE, IC1, IC2 }, x(t) ) );
```

$$EOM := \frac{1}{2}\cos\left(\frac{7}{10}\sqrt{10}\,t\right) - \frac{1}{14}\sin\left(\frac{7}{10}\sqrt{10}\,t\right)\sqrt{10}$$

As in all previous cases, the solution is most useful when written in the form x(t) = A*sin(omega*t+phi). Here this requires

```
> omega := 7/sqrt(10);
  evalf(omega);
```

$$\omega := \frac{7}{10}\sqrt{10}$$

$$2.213594362$$

```
> A := sqrt( (1/2)^2 + (-sqrt(10)/14)^2 );
  evalf( A );
```

$$A := \frac{1}{14}\sqrt{59}$$

$$.5486532677$$

```
> phi := arctan( 1/2, -sqrt(10)/14 );
```

$$\phi := -\arctan\left(\frac{7}{10}\sqrt{10}\right) + \pi$$

Thus, the equation of motion can be equivalently written as

> EOMa := A * sin(omega*t + phi);

$$EOMa := -\frac{1}{14}\sqrt{59}\,\sin\left(\frac{7}{10}\sqrt{10}\,t - \arctan\left(\frac{7}{10}\sqrt{10}\right)\right)$$

> EOMa := -1/14*59^(1/2)*sin(7/10*10^(1/2)*t-arctan(7/10*10^(1/2)));

The mass is in the equilibrium position when the displacement is zero. This occurs whenever the argument of sine is an integer multiple of Pi, that is

> EQ := solve(7/10*10^(1/2)*t-arctan(7/10*10^(1/2))=n*Pi , t);

$$EQ := -\frac{1}{7}\left(-\arctan\left(\frac{7}{10}\sqrt{10}\right) - n\,\pi\right)\sqrt{10}$$

> evalf(subs(n=0, EQ));

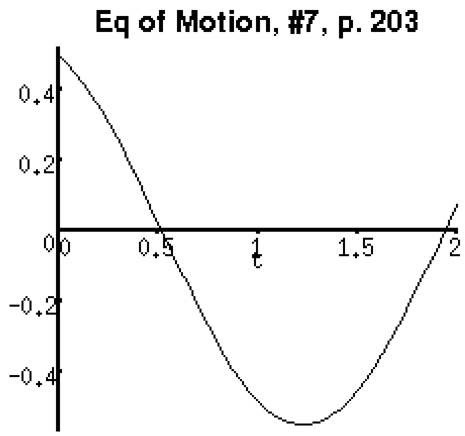
$$.5179290257$$

> evalf(subs(n=1, EQ));

$$1.937155922$$

The plot confirms that the first of these is the first time when the mass returns to the equilibrium position

> plot(EOMa, t=0..2, title='Eq of Motion, #7, p. 203');



END OF WORKSHEET

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