MATH 241 (Section 502)
Prof. Meade

University of South Carolina
Spring 2002

Final Exam
May 4, 2002

Name: ____________________
e-mail address: ____________________

Instructions:

1. There are a total of 9 problems on 11 pages. Check that your copy of the exam has all of the problems.
2. You must show all of your work to receive credit for a correct answer.
3. Calculators are not permitted (because they are not needed).
4. You may use anything written on one side of one $8\frac{1}{2}$"×11" sheet of paper.
5. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	9	
2	12	
3	10	
4	12	
5	12	
6	12	
7	12	
8	12	
9	9	
Total	100	

Have A Great Summer!

1. (9 points) Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{k}$, and $\mathbf{c} = \langle 1, 2, -1 \rangle$. Find each of the following quantities. *Hint*: Think before you compute!

   (a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

   (b) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c})$

   (c) $\mathbf{c} \cdot \mathbf{c} - |\mathbf{c}|^2$

2. (9 points)

   (a) What is the direction of the line $x = 2 - 3t$, $y = 3t$, $z = 2 - t$?
   What is a point on the line?
   NOTE: Be sure to clearly label the two answers.

   (b) Find symmetric equations for the line through $(6, 1, -3)$ and $(-2, 2, 3)$.

   (c) Find an equation for the plane containing the lines $x = 3t$, $y = 1 + t$, $z = 4 + t$ and $x = 3 - 2s$, $y = 2$, $z = 5 + 4s$.

3. (10 points) Let C be the parametric curve with position vector $\mathbf{r}(t) = (1 - t^2)\mathbf{i} + 2t\mathbf{j}$.

(a) Find the time(s) t when the curve passes through the point $P(0, 2)$.

(b) Find the unit tangent vector $\mathbf{T}$ at $P(0, 2)$.

(c) *Without finding the curvature*, find the tangential and normal components, a_T and a_N, of acceleration $\mathbf{a}$ at $P(0, 2)$.

4. (15 points) Let $f(x, y) = e^{-x}\sin(xy)$. Find

(a) $\nabla f(2, \frac{\pi}{4})$

(b) find the direction where f is increasing most rapidly at $(2, \frac{\pi}{4})$.

(c) f_{yy}

(d) $\frac{df}{dt}$ where $x = t$ and $y = \frac{\pi}{t}$. Note: Your answer should be expressed in terms of t.

5. (9 points) Let $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$.

(a) Show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along every straight line.

(b) Show that $f(x,y) \to \dfrac{a}{1+a^2}$ as $(x,y) \to (0,0)$ along the parabola $y = ax^2$.

(c) What does this tell you about $\lim\limits_{(x,y)\to(0,0)} f(x,y)$?

6. (12 points) A recent challenge problem posted in the USC Department of Mathematics asked for a proof of the statement that

$$(a+\frac{1}{a})^2+(b+\frac{1}{b})^2 \geq \frac{25}{2}$$

when a and b are positive and $a+b=1$. My "solution" was "ask a MATH 241 student". I now want you to prove me correct!

My solution is based on the observation that the statement can be proved by solving the constrained optimization problem

$$\begin{array}{ll} \text{minimize} & f(a,b)=(a+\frac{1}{a})^2+(b+\frac{1}{b})^2 \\ \text{subject to} & a+b=1 \ (a,\, b \geq 0) \end{array}$$

(a) Write the Lagrange equations for the problem found in (a).
NOTE: Do not attempt to solve these equations!

(b) Based on your intuition, which — if any — of the following points (a, b) do you think will be most likely to produce the minimizer for the constrained optimization problem? Think about this problem. What can you say about $f(a, b)$ and $f(b, a)$? (Explain — in words — the reason for your answer.)

i.	$(0, 1)$	ii.	$(\frac{1}{10}, \frac{9}{10})$	iii.	$(\frac{1}{5}, \frac{4}{5})$	iv.	$(\frac{1}{4}, \frac{3}{4})$	v.	$(\frac{3}{10}, \frac{7}{10})$
vi.	$(\frac{1}{3}, \frac{2}{3})$	vii.	$(\frac{2}{5}, \frac{3}{5})$	viii.	$(\frac{1}{2}, \frac{1}{2})$	ix.	$(\frac{3}{5}, \frac{2}{5})$	x.	$(\frac{2}{3}, \frac{1}{3})$
xi.	$(\frac{7}{10}, \frac{3}{10})$	xii.	$(\frac{3}{4}, \frac{1}{4})$	xiii.	$(\frac{4}{5}, \frac{1}{5})$	xiv.	$(\frac{9}{10}, \frac{1}{10})$	xv.	$(1, 0)$

(c) Which point identified in (b) satisfies the Lagrange equations for this problem?
HINT: If you think about the problem you can greatly reduce — but not eliminate — the work for this part of the problem.
NOTE: The check can be done without fully evaluating the expressions in the Lagrange equations. It helps to eliminate the Lagrange multiplier from the Lagrange equations, but do not do anything more to solve the Lagrange equations. Be patient and think!

(d) What is the minimum value for $f(a, b)$ and where does it occur? Does your answer prove the original statement is true? (Explain.)

7. (12 points)

   (a) Evaluate $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 2xy^2 \, dx \, dy$.

   (b) Sketch the domain of integration for the iterated integral $\int_0^1 \int_y^{\sqrt{y}} f(x,y) \, dx \, dy$.

   (c) Interchange the order of integration in the integral in (b).

8. (12 points) Write the triple iterated integral for the volume of a sphere of radius a in

   (a) Cartesian coordinates

   (b) cylindrical coordinates.

   (c) spherical coordinates.

(d) Evaluate one of the three triple iterated integrals found above.

9. (9 points) Consider the line integral $\int_C y^2\, dx + 2xy\, dy$ where C is the portion of the curve $x = 2y^2$ from $(0,0)$ to $(2,1)$.

(a) Is this line integral independent of path? (Explain.)

(b) Use any valid method to evaluate the line integral.

(c) Briefly list another method that could be used to evaluate this line integral. Why did you choose the method you used in (b) and not this method? (In hindsight, do you wish you would have used the method in (b) instead of the one you used in (a)?)