MAPLE WORKSHEET FOR THE ANIMATED DISPLAY OF PARTIAL SUMS OF POWER SERIES

Math 142 -- Spring 1996

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WWW URL: http://www.math.sc.edu/~meade/math142/worksheets/pow-series.ms

This worksheet is designed with the purpose of illustrating the convergence set for a power series. The convergence of a power series in terms of the convergence of the sequence of partial sums (no different from regular infinite series). The only difference is that the convergence may depend on the value of x (or whatever the independent variable happens to be called).

You should see, as a result of this worksheet, that the CONVERGENCE SET is an interval (possibly including the entire real line). That is, if the series converges for two values of x, then it must converge for all intermediate values as well. (Why should this be expected?) A further observation should be that the convergence set is symmetric. The issue of convergence at the endpoints of the interval of convergence can be difficult to see graphically. Fortunately, it’s not difficult to consider the (no more than) two special points.

For each example you need to select both the number of partial sums to plot and the interval on which they should be plotted. Select these parameters so that the convergence set is easily identified.

```
> restart;
> with(plots):
> par_sum := proc( TERM, INDEX:name=range )
    local k, lo, hi, n;
    k := lhs(INDEX);
    lo := op(1,rhs(INDEX));
    hi := op(2,rhs(INDEX));
    seq( sum(TERM, k=lo..n), n=lo..hi )
  end:
> par_sum_plot := proc( TERM, INDEX:name=range, DOMAIN:name=range )
    local opts;
    opts := op( subsop(1=NULL,2=NULL,3=NULL, [args] ));
    display( map( plot, [ par_sum( TERM, INDEX ) ], DOMAIN), opts )
  end:
>
```

Two special commands are created to facilitate this study. The power series is assumed to be

```
> Sum( a[n](x-x0)^n, n=lo..infinity );
```

$$\sum_{n=lo}^{\infty} a_n (x - x0)^n$$

where lo is the index of the first term of the series (often 0 or 1). The first N partial sums of the series can be obtained with the par_sum command. For example:

```
> par_sum( x^n/n!, n=0..3 );
```

$$1,\ 1 + x,\ 1 + x + \frac{1}{2}x^2,\ 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

```
>
```

Plots of these partial sums can be obtained using the par_sum_plot command. This requires a third argument, the interval on which the partial sums are to be plotted. For example:

```
> par_sum_plot( x^n/n!, n=0..3, x=-2..1 );
>
```

These plots can become quite cluttered. An animated display is much less cluttered, and often provides a clear demonstration of the convergence of the partial sums. All that is needed is to specify insequence=true as an optional fourth argument to par_sum_plot. Other optional arguments, e.g. title and view, and also be specified. For example:

```
> par_sum_plot( x^n/n!, n=0..3, x=-2..1,  insequence=true );
>
```

EXAMPLE 1

The constant a can be factored out of the sum without changing the convergence of the series (unless a=0, which reduces the problem to a trivial case). Thus, to keep matters simple, let's assume a = 1.

Each term of the series is given by

```
> TERM := x^n;
```

$$TERM := x^n$$

The first three partial sums of this series are

```
> par_sum( TERM, n=0..3 );
```

$$1,\ 1+x,\ 1+x+x^2,\ 1+x+x^2+x^3$$

Note that these partial sums are simply polynomials, each with a higher degree than the previous partial sum.

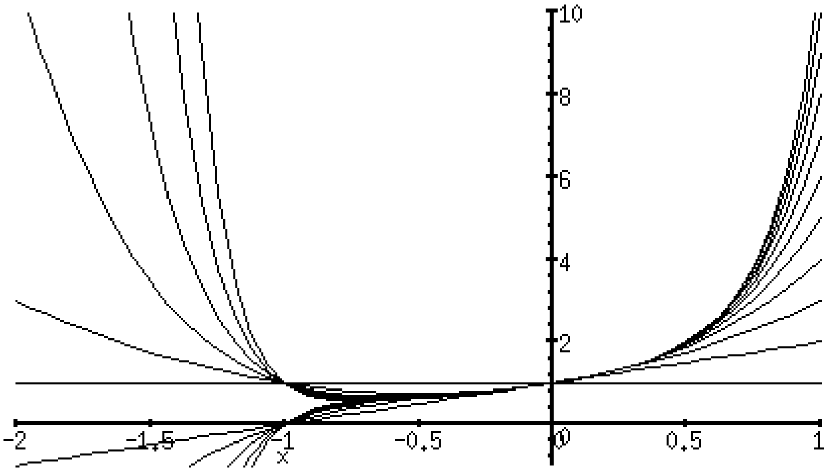
Does this series remind you of any that we've seen previously? Where (what values of x) do you expect this series to converge?

This convergence can be demonstrated graphically. Let's plot each partial sum individually. What trends do you notice in these plots?

```
> par_sum_plot( TERM, n=0..10, x=-2..1, insequence=true, view=[-2..1,-1..10] );
```

Another way to view the partial sum is to superimpose the partial sums in one plot. Does this help confirm the observations made from the animation? Is this consistent with what you expected to see for this problem? (Explain.)

```
> par_sum_plot( TERM, n=0..10, x=-2..1, view=[-2..1,-1..10] );
```




```
>
```

EXAMPLE 2

```
> TERM := x^n/(n+1)/2^n;
```

$$TERM := \frac{x^n}{(n+1)\,2^n}$$

```
> par_sum( TERM, n=0..3 );
```

$$1,\ 1+\frac{1}{4}x,\ 1+\frac{1}{4}x+\frac{1}{12}x^2,\ 1+\frac{1}{4}x+\frac{1}{12}x^2+\frac{1}{32}x^3$$

```
> par_sum_plot( TERM, n=0..??, x=??...??, insequence=true );
> par_sum_plot( TERM, n=0..??, x=??..?? );
>
```

EXAMPLE 3

```
> TERM := x^n/n!;
```

$$TERM := \frac{x^n}{n!}$$

```
> par_sum( TERM, n=0..3 );
```

$$1,\ 1+x,\ 1+x+\frac{1}{2}x^2,\ 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$$

```
> par_sum_plot( TERM, n=0..??, x=??...??, insequence=true );
> par_sum_plot( TERM, n=0..??, x=??..?? );
>
```

EXAMPLE 4

```
> TERM := n!*x^n;
```

$$TERM := n!\,x^n$$

```
> par_sum( TERM, n=0..3 );
```

$$1,\ 1+x,\ 1+x+2\,x^2,\ 1+x+2\,x^2+6\,x^3$$

```
> par_sum_plot( TERM, n=0..50, x=-1...1, insequence=true );
> par_sum_plot( TERM, n=0..10, x=-1..1 );
>
```

EXAMPLE 6

Note that the book is not quite right on this problem. Look at the first term of the series: ln(1)=0.

```
> TERM := ln(n)/n/3^n*(x+2)^n;
```

$$TERM := \frac{\ln(n)\,(x+2)^n}{n\,3^n}$$

```
> par_sum( TERM, n=1..3 );
```

$$0,\ \frac{1}{18}\ln(2)\,(x+2)^2,\ \frac{1}{18}\ln(2)\,(x+2)^2+\frac{1}{81}\ln(3)\,(x+2)^3$$

```
> par_sum_plot( TERM, n=1..10, x=-6...4, insequence=true );
> par_sum_plot( TERM, n=1..10, x=-6..4 );
>
```

Extra Credit - Fourier Series (due before April 22, 1996, but the last day for Maple help is April 15)

The text for Section 11.6 begins with an example of a series that is NOT a power series. (Why isn't this a power series?)

Convergence of this series is determined in exactly the same way as for all other series -- convergence of partial sums.

Investigate the convergence of this series, that is: does the series converge? for what values of x? Can you identify the limit of this series?
Be sure that any plot you produce clearly illustrates your conclusions. (Be careful when selecting the domain and number of partial sums to plot.)

Then, answer the same questions for the series

```
> Sum( 2/Pi*(1-(-1)^n)/n*sin(n*x), n=1..infinity ), Sum( (1-(-1)^n)/n^2*cos(n*Pi*x), n=1..infinity );
```

$$\sum_{n=1}^{\infty} \left(2 \frac{\left(1 - (-1)^n\right) \sin(n\,x)}{\pi\,n} \right), \sum_{n=1}^{\infty} \frac{\left(1 - (-1)^n\right) \cos(n\,\pi\,x)}{n^2}$$

Identify the limit of each of these series.

Note: The Maple name for the constant pi is Pi (the capital P, and lower case i, are necessary).

```
>
```