>

MAPLE WORKSHEET FOR NUMERICAL INTEGRATION

Math 142 (Sections 1 and 2) -- Spring 1994

Prepared by Douglas B. Meade

Illustrates the Trapezoidal and Simpson Rules for numerical integration. This uses the trapezoid and simpson commands in the student library. The graphical displays are modelled after the leftbox, etc. routines in the student library. Also, the Taylor series-based numerical integration is illustrated.

```
> #read(`num-int.maple`):
> # Maple Procedures for Graphs to Accompany trapezoid and simpson
  #
  # Prepared by Douglas B. Meade (6 Mar 1994)

  with( student ):
  with( plots ):

  trapezoid_box := proc(F,rng)
    local x,i,n,a,b,f,tbox,tboxes,width,parsedargs,plotopts;
    parsedargs := readlib(`student/checkboxargs`)(args);
    x := parsedargs[1];
    a := parsedargs[2];
    b := parsedargs[3];
    n := parsedargs[4];
    plotopts := op(parsedargs[5]);
    if not type(F,procedure) then
      f := subs({`trapbox/_x`=x, `trapbox/_y`=F},proc(`trapbox/_x`) `trapbox/_y` end)
    else
      f := F
    fi;
    width := (b-a)/n;
    tbox := <[a,0,a,f(a),b,f(b),b,0]|f,a,b>;
    trapboxes := {seq(tbox(f,a+i*width,a+(i+1)*width),i=0..n-1)};
    plot(trapboxes union {f(x)},x=a..b,'style'='LINE',plotopts)
  end:

  simpson_box := proc(F,rng)
    local x,i,n,a,b,f,sbar,sbars,scurve,scurves,width,parsedargs,plotopts;
    parsedargs := readlib(`student/checkboxargs`)(args);
    x := parsedargs[1];
    a := parsedargs[2];
    b := parsedargs[3];
    n := parsedargs[4];
    plotopts := op(parsedargs[5]);
    if not type(F,procedure) then
      f := subs({`simpbox/_x`=x, `simpbox/_y`=F},
             proc(`simpbox/_x`) `simpbox/_y` end)
    else
      f := F
    fi;
    width := (b-a)/n;
    scurve := proc(f,a,m,b)
      f(a)*(x-m)*(x-b)/(a-m)/(a-b)
      + f(m)*(x-a)*(x-b)/(m-a)/(m-b)
      + f(b)*(x-a)*(x-m)/(b-a)/(b-m)
    end;
    scurves := {seq(
    plot(scurve(f,a+2*i*width,a+(2*i+1)*width,a+(2*i+2)*width),
         x=a+2*i*width..a+(2*i+2)*width, 'style'='LINE', plotopts),
              i=0..n/2-1)};
    sbar := <[a,0,a,f(a)]|f,a,b>;
    sbars := {plot( {seq(sbar(f,a+i*width),i=0..n)}, x=a..b, 'style'='LINE', plotopts )};
    display( scurves union sbars
       union {plot(f(x),x=a..b, 'style'='LINE', plotopts)})
  end:
```

```
mycheckargs := proc(A)
  local x,a,b,n,x0,i,plotopts;
  n := 4;
  x0 := 0;
  plotopts := NULL;
  i := 4;
  if nargs<3 then
    ERROR(‘expecting at least three arguments (f,x=a..b,x0)‘)
  fi;
  if type([args[1..2]],[algebraic,name=range]) then
    x := op(1,args[2]);
    a := op(1,op(2,args[2]));
    b := op(2,op(2,args[2]));
    if nargs>=3 then
      x0 := op(2,args[3]);
      i:=4;
      if nargs>=4 then
        if type(args[4],posint) then
          n:=args[4];
          i:=5;
        fi;
      fi;
      plotopts:=args[i..nargs]
    fi;
  else
    ERROR(‘first two arguments must be (f,x=a..b,x0)‘);
  fi;
  [x,a,b,x0,n,[plotopts]]
end:

Taylor_int := proc(F,rng)
  local f,x,a,b,P,parsedargs,plotopts;
  parsedargs := mycheckargs(args);
  x := parsedargs[1];
  a := parsedargs[2];
  b := parsedargs[3];
  x0 := parsedargs[4];
  n := parsedargs[5];
  plotopts := op(parsedargs[6]);
  if not type(F,procedure) then
    f := subs({‘Taylor/_x‘=x, ‘Taylor/_y‘=F},
            proc(‘Taylor/_x‘) ‘Taylor/_y‘ end)
  else
    f := F
  fi;
  P := convert( taylor(f(x), x=x0, n+1), polynom );
  print( Int(P,x=a..b) = evalf( int(P,x=a..b) ) );
  plot( {f(x), P}, x=a..b, ’style’=’LINE’, plotopts );
end:

Trapezoid := proc()
  print( trapezoid(args) = evalf(trapezoid(args)) );
  trapezoid_box(args);
end:

Simpson := proc()
  print( simpson(args) = evalf(simpson(args)) );
  simpson_box(args);
end:
```

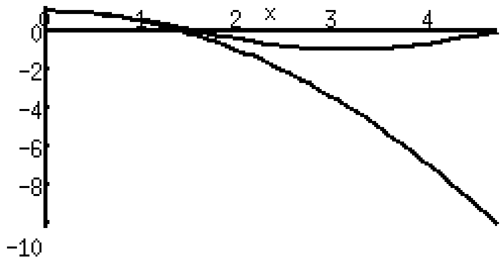
Let’s start with an example whose answer we know:

$$\int_{0}^{3/2\,\pi} \cos(x)\,dx = -1$$

One method of numerically computing this integral is to use a Taylor polynomial. For example, here’s the integral of the second-order Taylor polynomial of cos(x) centered at a=0:

> Taylor_int(cos(x), x=0..3*Pi/2, a=0, 2);

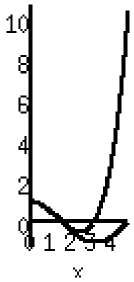
$$\int_{0}^{3/2\,\pi} 1 - \frac{1}{2}x^2\,dx = -12.72864166$$



It’s no more difficult to ask for higher-order approximations to the cosine function. Here’s the fourth-order apprximation:

> Taylor_int(cos(x), x=0..3*Pi/2, a=0, 4);

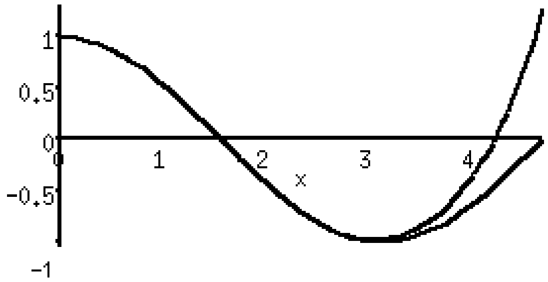
$$\int_{0}^{3/2\,\pi} 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\,dx = 6.63666653$$



... and the eighth-order Taylor polynomial of cos(x) at a=0:

> Taylor_int(cos(x), x=0..3*Pi/2, a=0, 8);

$$\int_0^{3/2\,\pi} 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 \, dx = -.44436593$$

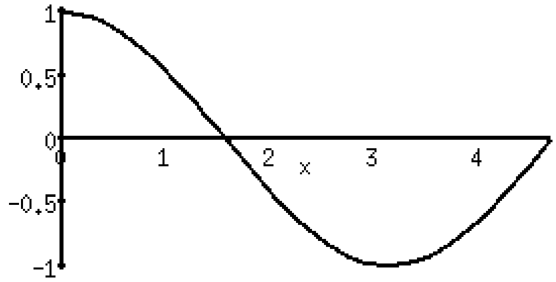


... and, finally, the 16th-order Taylor polynomial of cos(x) at a=0:

> Taylor_int(cos(x), x=0..3*Pi/2, a=0, 16);

$$\int_0^{3/2\,\pi} 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{3628800}x^{10} + \frac{1}{479001600}x^{12}$$

$$- \frac{1}{87178291200}x^{14} + \frac{1}{20922789888000}x^{16} \, dx = -.9999516972$$

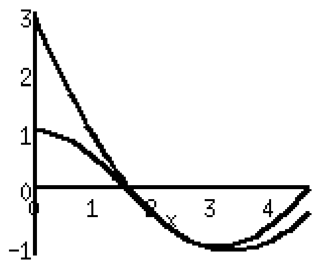


Would you have the patience to do this by hand?

Some questions need to be addressed before we accept this as a reasonable method for approximating integrals. First, how do the computed values depend on the base point, a. In the above examples a=0. Let's look at the same sequence, but with a=Pi/4.

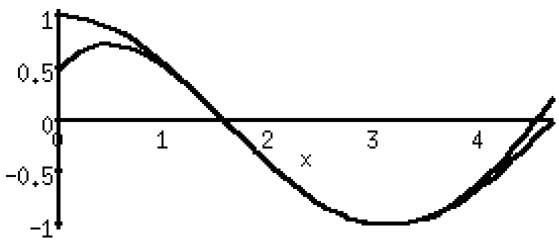
> Taylor_int(cos(x), x=0..3*Pi/2, a=3*Pi/4, 2);

$$\int_0^{3/2\,\pi} -\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}\left(x-\frac{3}{4}\pi\right)+\frac{1}{4}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^2 dx = -.248994446$$



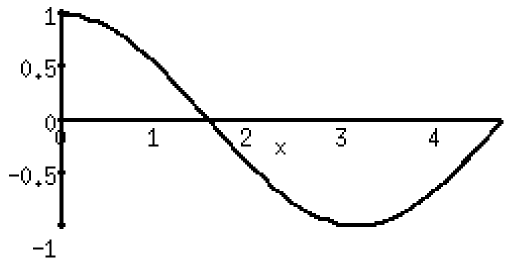
> Taylor_int(cos(x), x=0..3*Pi/2, a=3*Pi/4, 4);

$$\int_0^{3/2\,\pi} -\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}\left(x-\frac{3}{4}\pi\right)+\frac{1}{4}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^2+\frac{1}{12}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^3$$
$$-\frac{1}{48}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^4 dx = -1.104828242$$



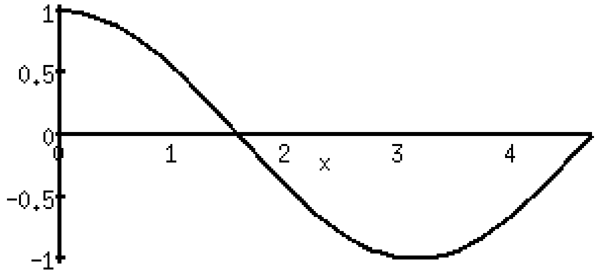
```
> Taylor_int( cos(x), x=0..3*Pi/2, a=3*Pi/4, 8 );
```

$$\int_{0}^{3/2\,\pi} -\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}\left(x-\frac{3}{4}\pi\right)+\frac{1}{4}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{2}+\frac{1}{12}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{3}-\frac{1}{48}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{4}-\frac{1}{240}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{5}+\frac{1}{1440}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{6}+\frac{1}{10080}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{7}-\frac{1}{80640}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{8}\,dx = -1.000424972$$



```
> Taylor_int( cos(x), x=0..3*Pi/2, a=3*Pi/4, 16 );
```

$$\int_0^{3/2\,\pi} -\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}\left(x-\frac{3}{4}\pi\right)+\frac{1}{4}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^2+\frac{1}{12}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^3$$
$$-\frac{1}{48}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^4-\frac{1}{240}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^5+\frac{1}{1440}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^6$$
$$+\frac{1}{10080}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^7-\frac{1}{80640}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^8$$
$$-\frac{1}{725760}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^9+\frac{1}{7257600}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{10}$$
$$+\frac{1}{79833600}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{11}-\frac{1}{958003200}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{12}$$
$$-\frac{1}{12454041600}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{13}+\frac{1}{174356582400}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{14}$$
$$+\frac{1}{2615348736000}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{15}$$
$$-\frac{1}{41845579776000}\sqrt{2}\left(x-\frac{3}{4}\pi\right)^{16}\,dx = \text{-1.000000000}$$



These values appear to be more accurate. But, note the more difficult computations that are necessary. Here the fact that values for sqrt(2) and Pi are needed is not good. In general, the values of the first 20 derivatives must be known before this final computation can be completed. For the cosine function this is not too bad, but think about an integrand that would be very messy to differentiate, e.g.

$$e^{(-x^2)}+x^2+2\sin(x^2+1)$$

We will return to this example shortly.

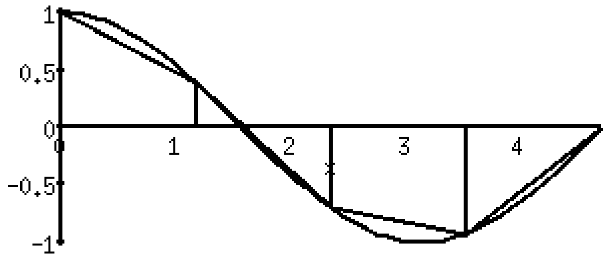
To be most effective, our numerical integration technique should be expressed ONLY in terms of values of the function, not its derivatives. The Trapezoidal and Parabolic (Simpson's) Rules have this property. Let's look at the first example, then turn to the messy problem.

Notice that the improvement in the approximations as more and more sub-intervals are used (i.e., as n increases) with the Trapezoid Rule. This is what is meant when we say the approximations CONVERGE to the exact value.

We start by subdividing the interval into 4 subintervals and 'connecting the dots'. This is the Trapezoid Rule with n=4:

> Trapezoid(cos(x), x=0..3*Pi/2, 4);

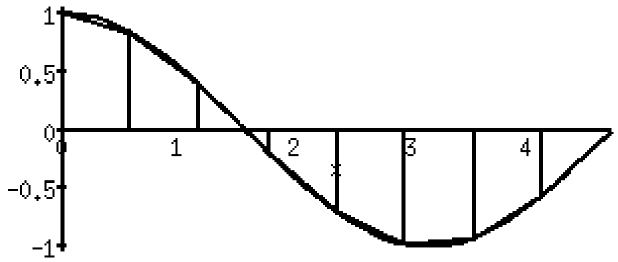
$$\frac{3}{16}\pi\left(1+2\left(\sum_{i=1}^{3}\cos\left(\frac{3}{8}i\pi\right)\right)\right) = -.8815735628$$



Let's try this with more subintervals.

> Trapezoid(cos(x), x=0..3*Pi/2, 8);

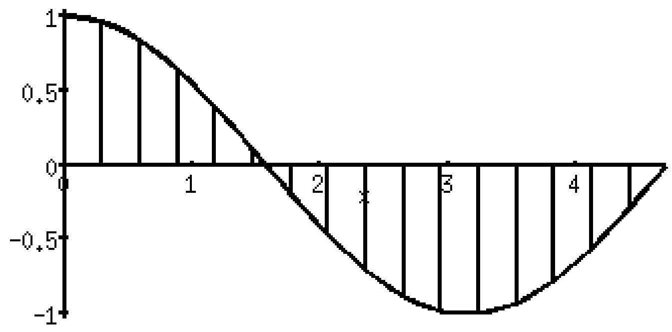
$$\frac{3}{32}\pi\left(1+2\left(\sum_{i=1}^{7}\cos\left(\frac{3}{16}i\pi\right)\right)\right) = -.9709165359$$



Note that with more subintervals there is less area between the integrand and its ‘‘piecewise linear approximation’’. This pattern continues as the number of subintervals increases. Here is n=16:

```
> Trapezoid( cos(x), x=0..3*Pi/2, 16 );
```

$$\frac{3}{64}\pi\left(1+2\left(\sum_{i=1}^{15}\cos\left(\frac{3}{32}i\,\pi\right)\right)\right)=-.9927608138$$

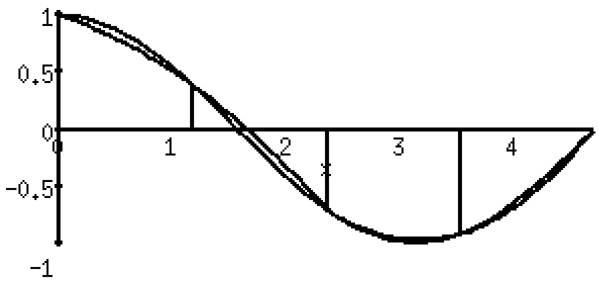


Even with 16 sub-intervals, the error is still relatively large: 0.0072. To improve the approximation of the integral it is necessary to improve the approximation of the integrand. One way to do this is to replace the line segments with parts of parabolas. This is Simpson's method. Let's see how it works, first with n=4:

> Simpson(cos(x), x=0..3*Pi/2, 4);

$$\frac{1}{8}\pi\left(1+4\left(\sum_{i=1}^{2}\cos\left(\frac{3}{8}(2i-1)\pi\right)\right)+2\left(\sum_{i=1}^{1}\cos\left(\frac{3}{4}i\pi\right)\right)\right)=$$

$$-1.012770132$$

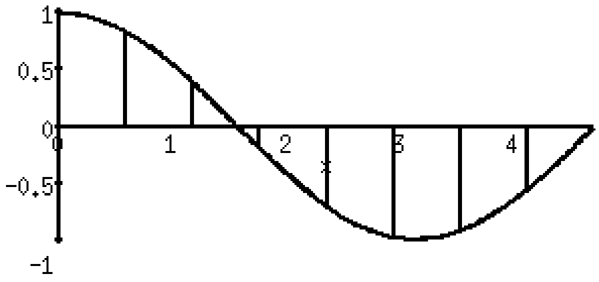


... then with n=8:

> Simpson(cos(x), x=0..3*Pi/2, 8);

$$\frac{1}{16}\pi\left(1+4\left(\sum_{i=1}^{4}\cos\left(\frac{3}{16}(2i-1)\pi\right)\right)+2\left(\sum_{i=1}^{3}\cos\left(\frac{3}{8}i\pi\right)\right)\right)=$$

$$-1.000697527$$

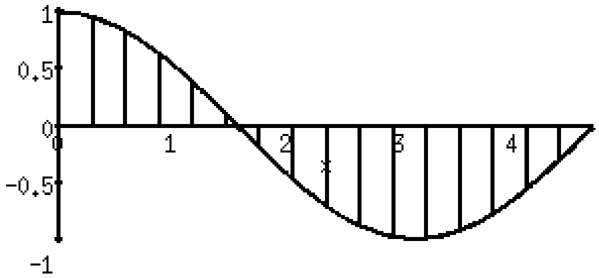


... and finally with n=16:

> Simpson(cos(x), x=0..3*Pi/2, 16);

$$\frac{1}{32}\pi\left(1+4\left(\sum_{i=1}^{8}\cos\left(\frac{3}{32}(2i-1)\pi\right)\right)+2\left(\sum_{i=1}^{7}\cos\left(\frac{3}{16}i\pi\right)\right)\right)=$$

$$-1.000042239$$



Notice the improved accuracy of these computations. With n=8, the error is on 0.0007. For n=16, the error is 0.00004 (so the integral two more digits of accuracy than the corresponding Trapezoid Rule).

Now, lets work on a problem whose exact answer we do NOT know:

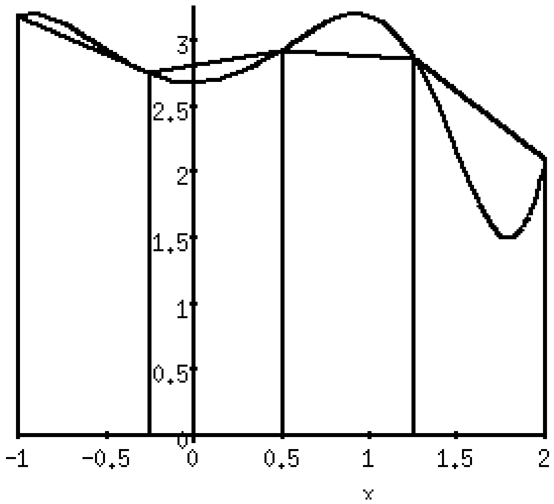
$$\int_{-1}^{2} e^{(-x^2)} + x^2 + 2\sin(x^2+1)\,dx$$

We will start with the Trapezoid Rule, then repeat the calculations for the Parabolic (Simpson's) Rule. Then, a final look at Taylor polynomials will show the problems that can arise with Taylor series.

To start, let's use a small number of subintervals, say n=4:

> Trapezoid(exp(-x^2) + x^2 + 2*sin(x^2+1), x=-1..2, 4);

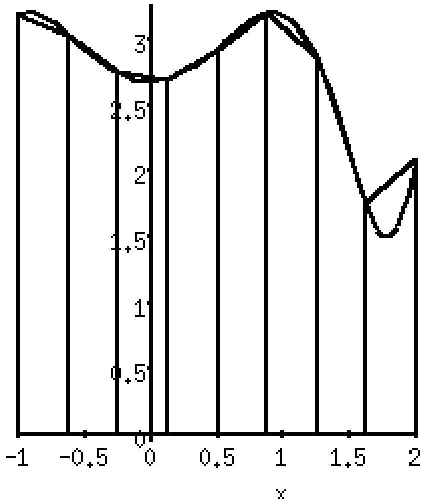
$$\frac{3}{8}e^{(-1)} + \frac{15}{8} + \frac{3}{4}\sin(2) + \frac{3}{4}\left(\sum_{i=1}^{3}\left(e^{\left(-(-1+3/4\,i)^2\right)} + \left(-1+\frac{3}{4}i\right)^2 + 2\sin\left(\left(-1+\frac{3}{4}i\right)^2+1\right)\right)\right) + \frac{3}{8}e^{(-4)} + \frac{3}{4}\sin(5) = 8.389458402$$



What do you expect to see as n increases? Here are n=8:

```
> Trapezoid( exp(-x^2) + x^2 + 2*sin(x^2+1), x=-1..2, 8 );
```

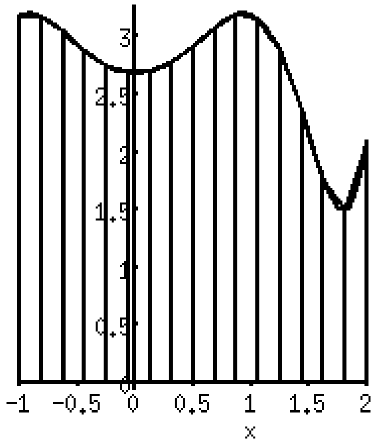
$$\frac{3}{16}\,e^{(-1)}+\frac{15}{16}+\frac{3}{8}\sin(2)$$
$$+\frac{3}{8}\left(\sum_{i=1}^{7}\left(e^{\left(-(-1+3/8\,i)^2\right)}+\left(-1+\frac{3}{8}\,i\right)^2+2\sin\left(\left(-1+\frac{3}{8}\,i\right)^2+1\right)\right)\right)$$
$$+\frac{3}{16}\,e^{(-4)}+\frac{3}{8}\sin(5)=8.200571423$$



... and n=16:

```
> Trapezoid( exp(-x^2) + x^2 + 2*sin(x^2+1), x=-1..2, 16 );
```

$$\frac{3}{32}\,e^{(-1)}+\frac{15}{32}+\frac{3}{16}\sin(2)+\frac{3}{16}\left(\sum_{i=1}^{15}\left(e^{\left(-(-1+3/16\,i)^2\right)}+\left(-1+\frac{3}{16}\,i\right)^2+2\sin\left(\left(-1+\frac{3}{16}\,i\right)^2+1\right)\right)\right)$$
$$+\frac{3}{32}\,e^{(-4)}+\frac{3}{16}\sin(5)=8.149687906$$



This finally is an answer that looks to be fairly accurate. Note that the errors are largest where the function is bending the most. (This fact also appears in the error estimate for the Trapezoid Rule, which depends on the second derivative of the integrand.)

This last observation suggests (just as in the previous example) that approximating the integrand with curved functions yield improved numerical accuracy without having to use so many subintervals. Quadratic functions (parabolas) are the next most convenient functions. The idea is to find the parabola passing through any three successive points.

Here are the pictures and numbers for this example. First with n=4 (i.e. two parabolas):

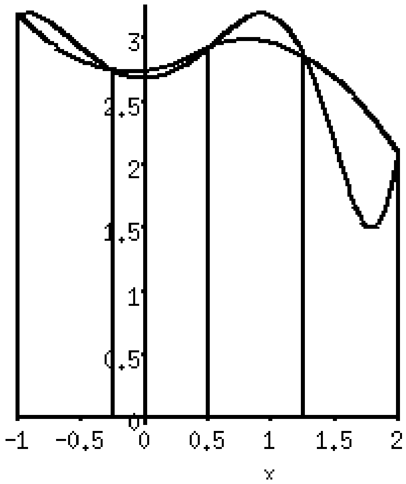
> Simpson(exp(-x^2) + x^2 + 2*sin(x^2+1), x=-1..2, 4);

$$\frac{1}{4}\,e^{(-1)}+\frac{5}{4}+\frac{1}{2}\sin(2)+\frac{1}{4}\,e^{(-4)}+\frac{1}{2}\sin(5)$$

$$+\left(\sum_{i=1}^{2}\left(e^{\left(-(-7/4+3/2\,i)^2\right)}+\left(-\frac{7}{4}+\frac{3}{2}\,i\right)^2+2\sin\left(\left(-\frac{7}{4}+\frac{3}{2}\,i\right)^2+1\right)\right)\right)$$

$$+\frac{1}{2}\left(\sum_{i=1}^{1}\left(e^{\left(-(-1+3/2\,i)^2\right)}+\left(-1+\frac{3}{2}\,i\right)^2+2\sin\left(\left(-1+\frac{3}{2}\,i\right)^2+1\right)\right)\right)$$

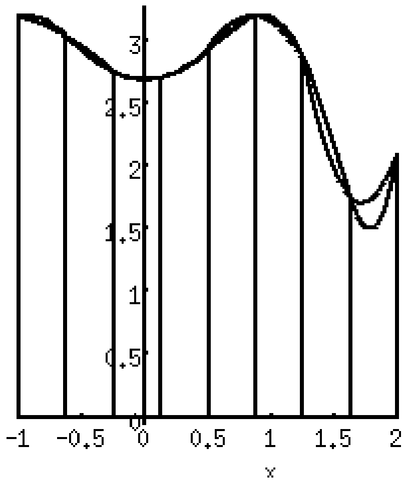
$$=8.400824177$$



... n=8 (4 parabolas):

```
> Simpson( exp(-x^2) + x^2 + 2*sin(x^2+1), x=-1..2, 8 );
```

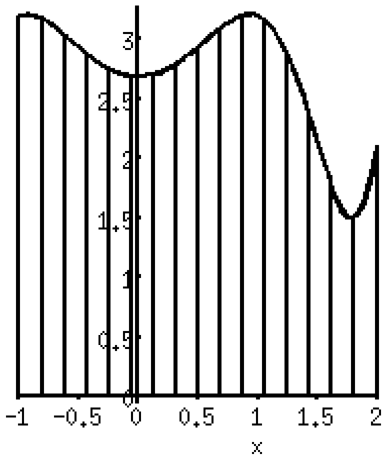
$$\frac{1}{8}\,e^{(-1)} + \frac{5}{8} + \frac{1}{4}\sin(2) + \frac{1}{8}\,e^{(-4)} + \frac{1}{4}\sin(5) + \frac{1}{2}$$
$$\left(\sum_{i=1}^{4}\left(e^{\left(-(-11/8+3/4\,i)^2\right)} + \left(-\frac{11}{8}+\frac{3}{4}\,i\right)^2 + 2\sin\left(\left(-\frac{11}{8}+\frac{3}{4}\,i\right)^2+1\right)\right)\right)$$
$$+\frac{1}{4}\left(\sum_{i=1}^{3}\left(e^{\left(-(-1+3/4\,i)^2\right)} + \left(-1+\frac{3}{4}\,i\right)^2 + 2\sin\left(\left(-1+\frac{3}{4}\,i\right)^2+1\right)\right)\right)$$
$$= 8.137609099$$



... and, finally, n=16 (8 parabolas):

> Simpson(exp(-x^2) + x^2 + 2*sin(x^2+1), x=-1..2, 16);

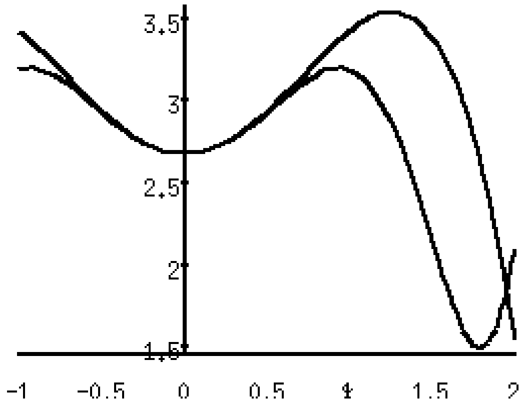
$$\frac{1}{16}\,e^{(-1)}+\frac{5}{16}+\frac{1}{8}\sin(2)+\frac{1}{16}\,e^{(-4)}+\frac{1}{8}\sin(5)+\frac{1}{4}\left(\sum_{i=1}^{8}\left(e^{\left(-\left(-\frac{19}{16}+3/8\,i\right)^2\right)}+\left(-\frac{19}{16}+\frac{3}{8}\,i\right)^2+2\sin\left(\left(-\frac{19}{16}+\frac{3}{8}\,i\right)^2+1\right)\right)\right)$$
$$+\frac{1}{8}\left(\sum_{i=1}^{7}\left(e^{\left(-(-1+3/8\,i)^2\right)}+\left(-1+\frac{3}{8}\,i\right)^2+2\sin\left(\left(-1+\frac{3}{8}\,i\right)^2+1\right)\right)\right)$$
$$=8.132726731$$



The Taylor Polynomials can also be used to approximate this integral. Once again we have to select the center for the polynomial. To keep life as simple as possible (another meaning for ASAP!), I will choose a=0:

> Taylor_int(exp(-x^2) + x^2 + 2*sin(x^2+1), x=-1..2, a=0, 4);

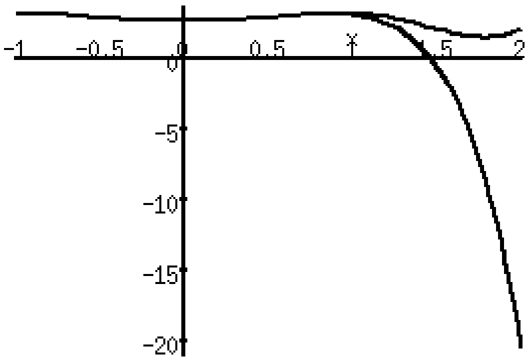
$$\int_{-1}^{2} 1 + 2\sin(1) + 2\cos(1)\,x^2 + \left(-\sin(1) + \frac{1}{2}\right)x^4\,dx = 9.036931244$$



> Taylor_int(exp(-x^2) + x^2 + 2*sin(x^2+1), x=-1..2, a=0, 6);

$$\int_{-1}^{2} 1 + 2\sin(1) + 2\cos(1)\,x^2 + \left(-\sin(1) + \frac{1}{2}\right)x^4 + \left(-\frac{1}{6} - \frac{1}{3}\cos(1)\right)x^6\,dx$$

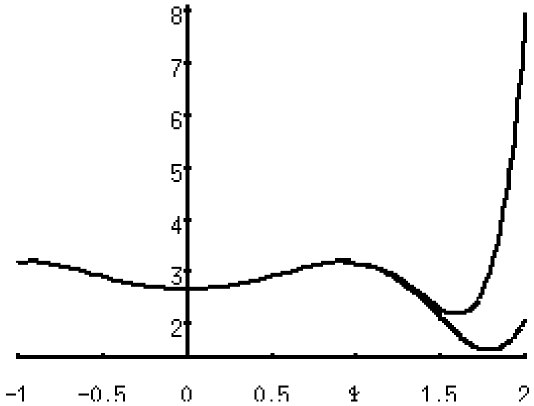
$$= 2.646502794$$



Note how poorly this function approximates the integrand and the corresponding error in the numerical approximation!

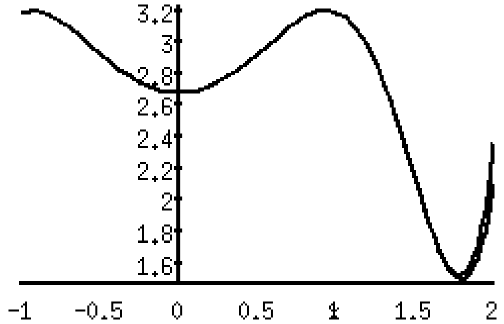
> Taylor_int(exp(-x^2) + x^2 + 2*sin(x^2+1), x=-1..2, a=0, 8);

$$\int_{-1}^{2} 1 + 2\sin(1) + 2\cos(1)\,x^2 + \left(-\sin(1) + \frac{1}{2}\right)x^4 + \left(-\frac{1}{6} - \frac{1}{3}\cos(1)\right)x^6 + \left(\frac{1}{24} + \frac{1}{12}\sin(1)\right)x^8\,dx = 9.018489972$$



> Taylor_int(exp(-x^2) + x^2 + 2*sin(x^2+1), x=-1..2, a=0, 16);

$$\int_{-1}^{2} 1 + 2\sin(1) + 2\cos(1)\,x^2 + \left(-\sin(1) + \frac{1}{2}\right)x^4 + \left(-\frac{1}{6} - \frac{1}{3}\cos(1)\right)x^6 + \left(\frac{1}{24} + \frac{1}{12}\sin(1)\right)x^8 + \left(-\frac{1}{120} + \frac{1}{60}\cos(1)\right)x^{10} + \left(-\frac{1}{360}\sin(1) + \frac{1}{720}\right)x^{12} + \left(-\frac{1}{5040} - \frac{1}{2520}\cos(1)\right)x^{14} + \left(\frac{1}{40320} + \frac{1}{20160}\sin(1)\right)x^{16}\,dx = 8.157018719$$



The problems disappear as n increases, but look at all the extra work that has to be done!

The value of this integral, correct to 8 decimal places, is

$$8.132702730$$

There are two major points you should be seeing in these examples:

1. Compare the results for the Trapezoidal and Parabolic Rules with the same value of n. The Parabolic Rule will be closer to the exact value. That is, the Parabolic Rule yields a better approximation than the Trapezoidal Rule.

2. Note how poorly the Taylor polynomials approximate the correct value. The explanation for this is simple: Taylor polynomials are most accurate very near the base point, a; an accurate approximation to the integral requires an approximation that is valid on the entire interval of integration.

The point is this: While all three sequences of approximate values converge to the same limit, the rate at which they approximate the value of the integral is very different. In numerical analysis, the point is to find a technique which yields an accurate approximation to the value with the least amount of computational work. Here, the Parabolic Rule yields the best results.