>

MAPLE WORKSHEET for SERIES (Section 11.2)

Math 142 -- Spring 1994

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This worksheet is intended to provide a graphical interpretation of sequences and their limits. The Series command provides the following information:

i) numerical values for the first N partial sums of the series;
ii) the sum of the series, if it converges;
iii) a plot of the first N partial sums of the series.

REMEMBER: Do NOT confuse the sequence of partial sums and the sequence formed by the terms of the series.

If you are interested, you can use Maple on the PCs in the public lab in PSC.

```
> with(plots):
```

The first step is to define the Maple procedures for displaying series (and sequences) in a convenient manner.

```
> Sequence := proc( expr, arg:name=range )
    local f, i, ii, S, var, lo, hi, m, M;
    var := lhs(arg); lo := op(1,rhs(arg) ); hi := op(2, rhs(arg) );
    f := unapply( expr, var );
    S := [seq( [i, evalf(f(i))], i=lo..hi )];
    m := min( 0, seq( op(2,s), s=S ) );
    M := max( 0, seq( op(2,s), s=S ) );
    print( S );
    print( Limit( f(var), var=infinity ) = limit( f(ii), ii=infinity) );
    plot( S, var=lo..hi, y=m..M, style=POINT );
  end:
Warning, ‘s‘ is implicitly declared local
```

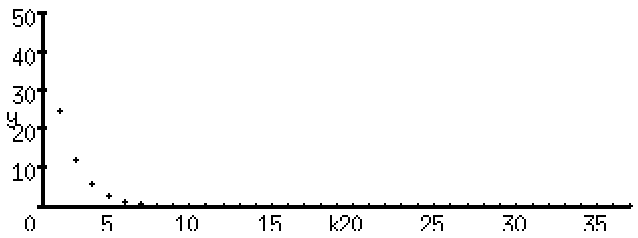
```
> Series := proc( expr, arg:name=range )
    local f, i, ii, j, P, s, var, lo, hi, m, M;
    var := lhs(arg); lo := op(1,rhs(arg) ); hi := op(2, rhs(arg) );
    f := unapply( expr, var );
    s := 0; P := NULL;
    for i from lo to hi do
      s := s+f(i);
      P := P, [i, evalf(s)];
    od;
    P := [P];
    m := min( 0, seq( op(2,p), p=P ) );
    M := max( 0, seq( op(2,p), p=P ) );
    print( P );
    print( Sum( f(var),var=lo..infinity ) = evalf( sum( f(ii), ii=lo..infinity ) ) );
    plot( P, var=lo..hi, y=m..M, style=POINT );
  end:
Warning, ‘p‘ is implicitly declared local
```

The first few examples focus on the distinction between the sequence of partial sums and the sequence of the terms of the series. First, here are the first few terms of the sequence formed by the terms of the series {a[k] }.

```
> Sequence( 100*(1/2)^k, k=1..37 );
```

$$[[1, 50.], [2, 25.], [3, 12.50000000], [4, 6.250000000], [5, 3.125000000], [6, 1.562500000], [7, .7812500000], [8, .3906250000], [9, .1953125000], [10, .09765625000], [11, .04882812500], [12, .02441406250], [13, .01220703125], [14, .006103515625], [15, .003051757813], [16, .001525878906], [17, .0007629394531], [18, .0003814697266], [19, .0001907348633], [20, .00009536743164], [21, .00004768371582], [22, .00002384185791], [23, .00001192092896], [24, .5960464478\ 10^{-5}], [25, .2980232239\ 10^{-5}], [26, .1490116119\ 10^{-5}], [27, .7450580597\ 10^{-6}], [28, .3725290298\ 10^{-6}], [29, .1862645149\ 10^{-6}], [30, .9313225746\ 10^{-7}], [31, .4656612873\ 10^{-7}], [32, .2328306437\ 10^{-7}], [33, .1164153218\ 10^{-7}], [34, .5820766091\ 10^{-8}], [35, .2910383046\ 10^{-8}], [36, .1455191523\ 10^{-8}], [37, .7275957614\ 10^{-9}]]$$

$$\lim_{k \to \infty} 100 \left(\frac{1}{2}\right)^k = 0$$

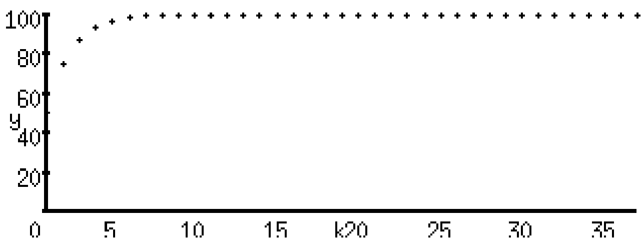


And, here are the first 50 partial sums (numerically and graphically). Notice that, in this case, both sequences converge, but to different limits.

```
> Series( 100*(1/2)^k, k=1..37 );
```

$$[[1, 50.], [2, 75.], [3, 87.50000000], [4, 93.75000000], [5, 96.87500000], [6, 98.43750000], [7, 99.21875000], [8, 99.60937500], [9, 99.80468750], [10, 99.90234375], [11, 99.95117188], [12, 99.97558594], [13, 99.98779297], [14, 99.99389648], [15, 99.99694824], [16, 99.99847412], [17, 99.99923706], [18, 99.99961853], [19, 99.99980927], [20, 99.99990463], [21, 99.99995232], [22, 99.99997616], [23, 99.99998808], [24, 99.99999404], [25, 99.99999702], [26, 99.99999851], [27, 99.99999925], [28, 99.99999963], [29, 99.99999981], [30, 99.99999991], [31, 99.99999995], [32, 99.99999998], [33, 99.99999999], [34, 99.99999999], [35, 100.00000000], [36, 100.00000000], [37, 100.00000000]]$$

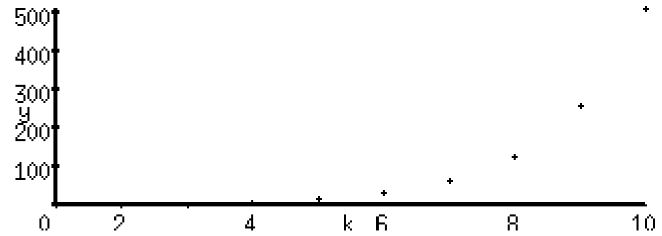
$$\sum_{k=1}^{\infty} \left(100 \left(\frac{1}{2} \right)^k \right) = 100.$$



Now, for some more geometric series. First, here's a geometric series with r>1 (what should happen here?):

> Sequence(2^(k-1), k=1..10);

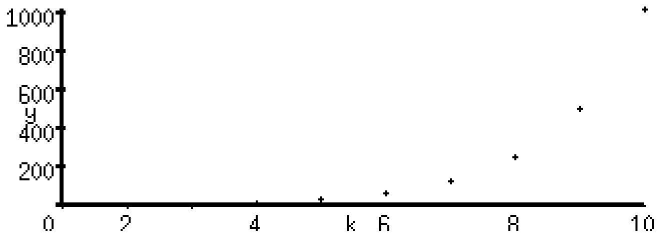
$$[[1, 1.], [2, 2.], [3, 4.], [4, 8.], [5, 16.], [6, 32.], [7, 64.], [8, 128.], [9, 256.], [10, 512.]]$$



$$\lim_{k \to \infty} 2^{(k-1)} = \infty$$

> Series(2^(k-1), k=1.. 10);

$$[[1, 1.], [2, 3.], [3, 7.], [4, 15.], [5, 31.], [6, 63.], [7, 127.], [8, 255.], [9, 511.], [10, 1023.]]$$



$$\sum_{k=1}^{\infty} 2^{(k-1)} = \infty$$

This is an example of a divergent series.

This geometric series is also divergent, but in a different way. Note that the sequence of terms alternates between 1 and -1; the sequence of partial sums alternates between 1 and 0.

> Sequence((-1)^(k-1), k=1..10);

$$[[1, 1.], [2, -1.], [3, 1.], [4, -1.], [5, 1.], [6, -1.], [7, 1.], [8, -1.], [9, 1.], [10, -1.]]$$

> Series((-1)^(k-1), k=1..10);

$$[[1, 1.], [2, 0], [3, 1.], [4, 0], [5, 1.], [6, 0], [7, 1.], [8, 0], [9, 1.], [10, 0]]$$

These two examples illustrate the n{th} Term Test for the DIVERGENCE of a series. Since the terms do NOT converge to 0, the series MUST DIVERGE.

Think about what the n{th} Term Test is saying. A series converges when you can add an infinite number of numbers to obtain a FINITE answer. If the individual terms do not become small (converge to 0), the sequence of partial sums will keep changing by this (not small) amount and so cannot be converging to a finite number.

The converse of the n{th} Term Test does NOT apply. Just because the terms converge to 0 does NOT IMPLY the series converges. The classical counter-example is the Harmonic Series:

```
> Sequence( 1/n, n=1..50 );
> Series( 1/n, n=1..40 );
```

The Alternating Harmonic Series is a different story. Here the terms converge to 0, and the series converges. Notice how the partial sums bounce back and forth. With each partial sum, the amplitude of the oscillation decreases; ultimately, the sequence of partial sums does converge (but not to a nice value).

```
> Series( (-1)^n/n, n=1..40 );
```

What do you think will happen with this series? Why do I start with k=3, not k=0?

```
> Series( 2/(k-1)/k, k=3..85 );
>
```

SUMMARY:

There have been three main points to this demonstration.

1. Notice the distinction between the sequence of partial sums and the sequence of terms of the series. The partial sums are difficult to find, in general. We will be developing techniques which permit us to make a conclusion about the sequence of partial sums (and
hence the series) while examining only the terms of the sequence. Please, work hard to understand this difference.

2. The n{th} Term Test is a simple test that enables us to examine the terms of the series to determine when some series diverge. This should be the first test that you apply to any series. Here's another statement of the n{th} Term Test:
   if lim a[n] <> 0 then the series MUST DIVERGE
   if lim a[n] = 0 then the series MAY CONVERGE (but some other test will have to be used to decide)

3. Geometric series converge only when the radius satisfies: -1 < r < 1.

```
>
```