MATH 142 (Section 501)
Prof. Meade

University of South Carolina
Fall 2003

Exam 4
November 25, 2003

Name: Key
SS #:

Instructions:

1. There are a total of 3 problems (including the Extra Credit problem) on 5 pages. Check that your copy of the exam has all of the problems.
2. *All work must be shown* to receive credit for a correct answer. (A brief description of your logic is also acceptable.)
3. Questions that ask for power series can be answered either by providing an infinite sum or by listing the first four terms with non-zero coefficients followed by an ellipsis (...).
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
5. *No calculators!* If you believe you need to use a calculator you are doing something wrong!!

Problem	Points	Score
1	32	
2	40	
3	28	
Total	100	

Happy Thanksgiving!

1. (32 points)

(a) [8 points] Which of the following infinite series is a power series expansion for $f(x) = \ln(1+x)$?
Explain how you made your decision.

i. $\sum_{n=1}^{\infty} \frac{x^n}{n}$

ii. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

iii. $\sum_{n=1}^{\infty} x^n$

iv. $\sum_{n=1}^{\infty} (-1)^{n+1} x^n$

v. $\sum_{n=1}^{\infty} n x^n$

vi. $\sum_{n=1}^{\infty} (-1)^{n+1} n x^n$

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x \sum_{n=0}^{\infty} (-t)^n\,dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n\,dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

(b) [8 points] Use your answer in (1a) to find the value of $f^{(10)}(0)$. (It is ok to leave factorials in your answer.) We know $\ln(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

Thus $\frac{f^{(n)}(0)}{n!} = (-1)^{n+1} \frac{1}{n}$. (for all n)

For $n=10$: $\frac{f^{(10)}(0)}{10!} = (-1)^{11} \frac{1}{10} = \frac{-1}{10}$ so $f^{(10)}(0) = \frac{-1}{10} \cdot 10! = -9!$

(c) [4 points] What is the *interval* of convergence of your answer in (1a)?

The geometric series has radius of convergence $r=1$. $\boxed{\text{interval: } (-1, 1]}$

The same is true for $\ln(1+x)$.

At endpoints: $x=1$: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges (alt. harm.) & $x=-1$: $\sum_{n=1} (-1)^{n+1} \frac{(-1)^n}{n} = \sum_{n=1} \frac{-1}{n}$ diverges.

(d) [8 points] Use your answer in (1a) to find the power series expansion for $g(x) = \ln(1-4x^2)$.

$$g(x) = \ln(1-4x^2) = \ln(1+(-4x^2)) = f(-4x^2)$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-4x^2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-1)^n 4^n x^{2n}}{n}$$
$$= \sum_{n=1}^{\infty} -\frac{4^n}{n} x^{2n}.$$

(e) [4 points] What is the *radius* of convergence of your answer in (1c)?

The power series for $g(x)$ converges when $|-4x^2| < 1$

$x^2 < 1/4$

$|x| < 1/2$

so the radius of convergence is $r = 1/2$.

2. (36 points)

(a) [7 points] What are the power series expansions for e^x and e^{-x}?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + \cdots$$

(b) [7 points] Use your power series in (2a) to obtain the power series for $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$.

$$\begin{aligned}\sinh x &= \tfrac{1}{2}\left(e^x - e^{-x}\right)\\ &= \tfrac{1}{2}\Big(\Big(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots\Big)\\ &\qquad - \Big(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + \cdots\Big)\Big)\\ &= x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\end{aligned}$$

(c) [4 points] What is the radius of convergence for the power series in (2b)?

The series for the exponential functions have radius of convergence $r = \infty$; the same is true of the series for sinh.

(d) [7 points] Use the power series in (2a) to obtain the power series for $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$?

$$\begin{aligned}\cosh x &= \tfrac{1}{2}\Big(\Big(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots\Big)\\ &\qquad + \Big(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} + \cdots\Big)\Big)\\ &= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\end{aligned}$$

(e) [7 points] Use the power series in (2b) to obtain another power series for $\cosh(x) = \frac{d}{dx}\sinh(x)$?

$$\begin{aligned}\cosh x = \frac{d}{dx}\sinh x &= \frac{d}{dx}\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right)\\ &= 1 + \frac{3x^2}{6} + \frac{5x^4}{120} + \frac{7x^6}{5040} + \cdots\\ &= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\end{aligned}$$

(f) [4 points] Should the answers to (2d) and (2e) agree or disagree? (Do they?)

Because both answers are power series (at $a=0$) for the same function, these power series should the same. They are the same!

3. (28 points) In this problem three different approximations will be found for $\int_{-2}^{4} \frac{\sin x}{x}\,dx$. The plot and table on the following page will be useful at various times throughout this problem.

(a) [6 points] Show that the error made when this integral is approximated by the Trapezoidal Rule with 10 subintervals is smaller than 0.06.

NOTE: Recall that $E_n = -\frac{(b-a)^3}{12n^2} f''(c)$ for some value of c between a and b.

$$|E_{10}| = \left| \frac{-(b-a)^3}{12\cdot 10^2} f''(c) \right| = \frac{6^3}{12\cdot 10^2} |f''(c)| \le \frac{6^3}{12\cdot 10^2}\cdot\frac{1}{3} = \frac{6}{100} = 0.06.$$

(note that this is in the table as well)

From the graph: $|f''(c)| \le \frac{1}{3}$ for $-2 \le x \le 4$

(b) [6 points] How large must n be to ensure that the Simpson's (Parabolic) Rule approximates the integral with an error E_n satisfying $|E_n| \le 0.0005$?

NOTE: Recall that $E_n = -\frac{(b-a)^5}{180n^4} f^{(4)}(c)$ for some value of c between a and b.

last column in table.

$$|E_n| = \left| -\frac{6^5}{180n^4} f^{(4)}(c) \right| = \frac{6^5}{180n^4} |f^{(4)}(c)| \le \frac{6^{\cancel{5}\,3}}{\cancel{180}_{5}\, n^4}\left(\frac{1}{5}\right) = \frac{6^3}{5^2 n^4} = \frac{216}{25n^4} \le 0.0005$$

From the graph: $|f^{(4)}(c)| \le \frac{1}{5}$ for $-2 \le x \le 4$. From the table, this is true when $n \ge 20$.

(c) [6 points] Find the Maclaurin polynomial of order 3 for the integrand of the integral.

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + \cdots$$

so $P_3(x) = 1 - \frac{x^2}{6}$.

(d) [6 points] Use your answer in (3c) to approximate the definite integral. (Leave your answer as a rational number.)

$$\int_{-2}^{4} \frac{\sin x}{x}\,dx = \int_{-2}^{4} P_3(x) + R_3(x)\,dx = \int_{-2}^{4} P_3(x)\,dx + \int_{-2}^{4} R_3(x)\,dx$$

$$\int_{-2}^{4} P_3(x)\,dx = \int_{-2}^{4} 1 - \frac{x^2}{6}\,dx = \left(x - \frac{x^3}{18}\right)\Big|_{-2}^{4} = \left(4 - \frac{4^3}{18}\right) - \left(-2 - \frac{(-2)^3}{18}\right) = 4 - \frac{64}{18} + 2 - \frac{8}{18} = 2$$

(e) [4 points] Obtain a bound for the absolute value of the error, $|R_3(x)|$, that is valid for $-2 \le x \le 4$.

$$|R_3(x)| = \left| \frac{f^{(4)}(c)}{4!} x^4 \right| \le \frac{1/5}{4!}(4)^4 = \frac{4^4}{5\cdot 4!} = \frac{4^3}{5\cdot 3!} = \frac{64}{30} = \frac{32}{15}$$

Note that this ~~error~~ bound is much larger than it actually is.

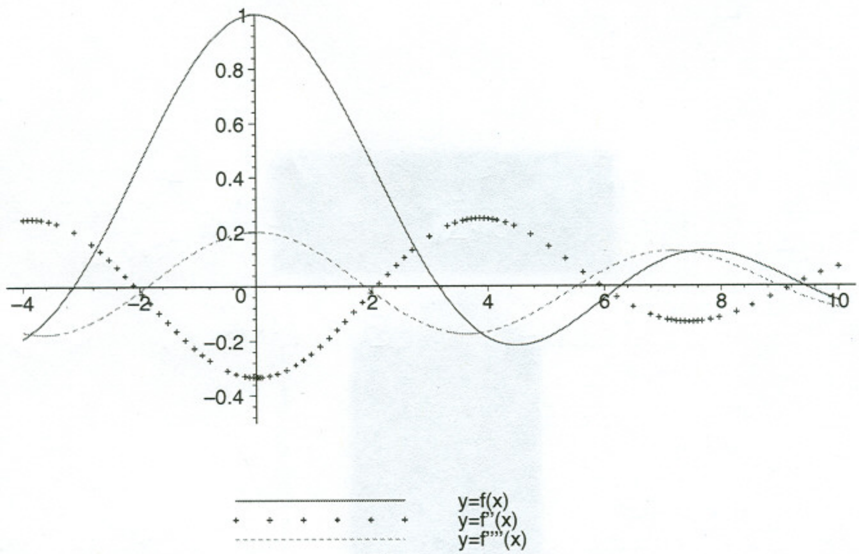
(f) [4 points] Use the bound found in (3f) to estimate the error in the approximation to the definite integral found in (3d).

Using (e): the error in approximation in (d) ~~is bounded by~~ is $\int_{-2}^{4} R_3(x)\,dx$.

This can be bounded by $\left|\int_{-2}^{4} R_3(x)\,dx\right| \le \int_{-2}^{4} |R_3(x)|\,dx \le \int_{-2}^{4} \frac{32}{15}\,dx = \frac{32}{15}\cdot 6 = \frac{64}{5}$

or, using $\left|\int_{-2}^{4} R_3(x)\,dx\right| \le \int_{-2}^{4} |R_3(x)|\,dx \le \int_{-2}^{4} \frac{1/5}{4!} x^4\,dx = \frac{1}{5!}\,\frac{1}{5}x^5\Big|_{-2}^{4} = \frac{4^5 - (-2)^5}{5\cdot 5!} = \frac{4^5 + 2^5}{5\cdot 5!}$

$$= \frac{2^5(2^5+1)}{5\cdot 5!} = \frac{32\cdot 33}{5\cdot 120} = \frac{44}{25}$$ (better, but still not great)



for (a)

for(b)

n	$\frac{36}{10n^2}$	$\frac{36}{8n^2}$	$\frac{36}{6n^2}$	$\frac{216}{15n^4}$	$\frac{216}{20n^4}$	$\frac{216}{25n^4}$
10	0.0360000	0.0450000	0.0600000	0.0014400	0.0010800	0.0008640
20	0.0090000	0.0112500	0.0150000	0.0000900	0.0000675	0.0000540
30	0.0040000	0.0050000	0.0066667	0.0000178	0.0000133	0.0000107
40	0.0022500	0.0028125	0.0037500	0.0000056	0.0000042	0.0000034
50	0.0014400	0.0018000	0.0024000	0.0000023	0.0000017	0.0000014
60	0.0010000	0.0012500	0.0016667	0.0000011	0.0000008	0.0000007
70	0.0007347	0.0009184	0.0012245	0.0000006	0.0000004	0.0000004
80	0.0005625	0.0007031	0.0009375	0.0000004	0.0000003	0.0000002
90	0.0004444	0.0005556	0.0007407	0.0000002	0.0000002	0.0000001
100	0.0003600	0.0004500	0.0006000	0.0000001	0.0000001	0.0000001