MATH 142 (Section 501)
Prof. Meade

University of South Carolina
Fall 2003

Exam 3
November 5, 2003

Name: Key
SS #: ____________

Instructions:

1. There are a total of 4 problems (including the Extra Credit problem) on 6 pages. Check that your copy of the exam has all of the problems.
2. *All work must be shown* to receive credit for a correct answer. (A brief description of your logic is also acceptable.)
3. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
4. *No calculators!* If you believe you need to use a calculator you are doing something wrong!!

Problem	Points	Score
1	24	
2	24	
3	32	
4	20	
Extra Credit	10	
Total	100	

Good Luck!

1. (24 points) [6 points each] Evaluate each limit. Remember to indicate every application of l'Hôpital's Rule.

(a) $\displaystyle\lim_{x\to 0}\frac{\cos x}{x^2} = +\infty$ because $\displaystyle\lim_{x\to 0}\cos x = 1$

and $\displaystyle\lim_{x\to 0} x^2 = 0$ (from the right)

(b) $$\lim_{t\to 0^+}\underset{(0)(-\infty)}{\sqrt{t}\ln t} = \lim_{t\to 0^+}\frac{\ln t}{t^{-1/2}} \underset{\substack{\infty/\infty \\ l'H}}{=} \lim_{t\to 0^+}\frac{1/t}{-\frac{1}{2}t^{-3/2}} = \lim_{t\to 0^+} -2t^{1/2} = 0.$$

(c) $$\lim_{\cancel{x}\,t\to 0}\frac{\ln\left((1+t)^2\right)}{t} \underset{0/0}{=} \lim_{t\to 0}\frac{2\ln(1+t)}{t} \underset{\substack{0/0 \\ l'H}}{=} \lim_{t\to 0}\frac{2/(1+t)}{1} = \lim_{t\to 0}\frac{2}{1+t} = 2.$$

(d) $$\lim_{\theta\to 0}\frac{\sin\theta-\tan\theta}{\theta^2} \underset{\substack{0/0 \\ l'H}}{=} \lim_{\theta\to 0}\frac{\cos\theta-\sec^2\theta}{2\theta} \underset{\substack{0/0 \\ l'H}}{=} \lim_{\theta\to 0}\frac{-\sin\theta-2\sec\theta(\sec\theta\tan\theta)}{2}$$

$$= \lim_{\theta\to 0}\frac{-\sin\theta-2\frac{\sin\theta}{\cos^3\theta}}{2} = \frac{0}{2} = 0.$$

Note: There are many other ways to work this problem.

2. (24 points) [6 points each] Evaluate each definite integral.

(a) $\int_{-\infty}^{0} e^{2x}\,dx = \lim_{A\to-\infty} \int_A^0 e^{2x}dx = \lim_{A\to-\infty} \frac{1}{2} e^{2x}\Big|_A^0$

$= \lim_{A\to-\infty} \left(\frac{1}{2}e^0 - \frac{1}{2}e^A\right) = \frac{1}{2} - 0 = \frac{1}{2}.$

(b) $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{A\to\infty} \int_2^A \frac{1}{x(\ln x)^2}dx = \lim_{A\to\infty} \int_{\ln 2}^{\ln A} u^{-2}du$

$u = \ln x \quad x=2 \to u = \ln 2$

$du = \frac{1}{x}dx \quad x = A \to u = \ln A$

$= \lim_{A\to\infty} \left(-u^{-1}\Big|_{\ln 2}^{\ln A}\right) = \lim_{A\to\infty} -\frac{1}{\ln A} + \frac{1}{\ln 2}$

$= \frac{1}{\ln 2}$

(c) $\int_{-1}^{1} \frac{1}{1-x}\,dx = \lim_{t\to1^-} \int_{-1}^{t} \frac{1}{1-x}dx = \lim_{t\to1^-} -\ln(1-x)\Big|_{-1}^{t}$

$= \lim_{t\to1^-} -\ln(1-t) + \ln(2) = +\infty$

(d) $\int_0^\infty \frac{1}{x^{5/3}}\,dx = \lim_{A\to\infty} \int_0^A x^{-5/3}dx = \lim_{A\to\infty} \left(-\frac{3}{2}x^{-2/3}\Big|_0^A\right)$

$= \lim_{A\to\infty}$

3. (32 points) [8 points each] Determine if each series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt[3]{k}}$: Positive-term series is $\sum_{k=1}^{\infty}\frac{1}{k^{1/3}}$ (p-series, $p = 1/3 < 1$)

This series diverges, so the original series is not abs. conv.

This is an alternating series with $a_k = \frac{1}{k^{1/3}}$. These terms are positive and decreasing, so the series converges by the Alternating Series Test.

This series is conditionally convergent.

(b) $\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^k$ $\lim_{k\to\infty}(1+\frac{1}{k})^k = e$ (by def'n of e, or l'Hopital's Rule) $\neq 0$

This series diverges by the n^{th} Term Test.

(c) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{e^{k^2}}$ Ratio Test: $r = \lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{(k+1)/e^{(k+1)^2}}{k/e^{k^2}}$

$$= \lim_{k\to\infty}\frac{k+1}{k}\frac{e^{k^2}}{e^{(k+1)^2}} = \lim_{k\to\infty}\frac{k+1}{k}\frac{e^{k^2}}{e^{k^2+2k+1}}$$

$$= \lim_{k\to\infty}\frac{k+1}{k}\frac{1}{e^{2k+1}} = 1\cdot 0 = 0$$

This series converges absolutely by the Ratio Test.

(d) $\sum_{k=1}^{\infty}k^2\left(\frac{2}{3}\right)^k$ Ratio Test: $r = \lim_{k\to\infty}\frac{(k+1)^2(\frac{2}{3})^{k+1}}{k^2(\frac{2}{3})^k} = \lim_{k\to\infty}\frac{(k+1)^2}{k^2}\left(\frac{2}{3}\right)$

$$= 1\cdot\frac{2}{3} = \frac{2}{3} < 1$$

This series converges absolutely by the Ratio Test.

Extra Credit (10 points) The Cauchy distribution (with parameter $\theta = 0$) has probability density function

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad \text{for } -\infty < x < \infty.$$

(a) Verify that the Cauchy distribution (with parameter $\theta = 0$) is a probability density function.

$f(x) \geq 0$ because $1+x^2 \geq 1$ for all x.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = \lim_{B\to-\infty} \int_{B}^{0} \frac{1}{\pi}\frac{dx}{1+x^2} + \lim_{A\to\infty} \int_{0}^{A} \frac{1}{\pi}\frac{dx}{1+x^2}$$

$$= \lim_{B\to-\infty}\left(\frac{1}{\pi}\arctan 0 - \frac{1}{\pi}\arctan B\right) + \lim_{A\to\infty}\left(\frac{1}{\pi}\arctan A - \frac{1}{\pi}\arctan 0\right)$$

$$= \lim_{B\to-\infty} -\frac{1}{\pi}\arctan B + \lim_{A\to\infty}\frac{1}{\pi}\arctan A$$

$$= -\frac{1}{\pi}(-\pi/2) + \frac{1}{\pi}(\pi/2) = \frac{1}{2}+\frac{1}{2} = 1.$$

(b) Explain why $\int_{-\infty}^{\infty} xf(x)\,dx = \infty$.

(c) It does not make any sense to say the mean of this distribution is infinity. The remainder of this problem introduces the concepts needed to create a meaningful definition of the mean for the Cauchy distribution.

Let $A > 0$. Show that $\int_{-A}^{0} xf(x)\,dx = -\int_{0}^{A} xf(x)\,dx$.

$$\int_{-A}^{0} xf(x)dx = \int_{A}^{0}(-u)f(-u)(-du) = \int_{A}^{0} u f(-u)du = \int_{A}^{0} u f(u)du$$

$u = -x$
$du = -dx$

$$= -\int_{0}^{A} u f(u)du$$

$$= -\int_{0}^{A} xf(x)dx.$$

(d) The *Cauchy principal value* of the integral $\int_{-\infty}^{\infty} F(x)\,dx = \infty$ is defined to be

$$\lim_{A\to\infty}\int_{-A}^{A} F(x)\,dx$$

whenever this limit exists. Find the Cauchy principle value of $\int_{-\infty}^{\infty} xf(x)\,dx$.