MATH 141 (Section 5 & 6)
Prof. Meade

Exam 3
November 1, 2013

University of South Carolina
Fall 20139

Name: Key
Section: 005 / 006 (circle one)

Instructions:

1. There are a total of 6 problems (including the Extra Credit problem) on 7 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	15	
2	15	
3	~~20~~ 0	
4	20	
5	16	
6	14	
Total	~~100~~ 80	

Have a Great Weekend!

1. (15 points) Find the absolute maximum and absolute minimum values (including where they occur) for $f(x) = \dfrac{x}{x^2+1}$ on the interval $[0, 3]$.

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$ (defined everywhere)

$f'(x) = 0 \iff 1 - x^2 = 0$
$\iff x = \pm 1.$ (-1 is not in the interval)

x	$f(x)$	
0	$0/1 = 0$	$\leftarrow$ absolute min.
3	$\frac{3}{9+1} = \frac{3}{10}$	$\leftarrow$ ~~absolute~~
1	$\frac{1}{1+1} = \frac{1}{2}$	$\leftarrow$ absolute max

2. (15 points) Find each limit. Be sure to indicate the indeterminate form each time you use l'Hôpital's Rule.

(a) $\lim_{x\to 0} \dfrac{\cos(x) - 1 + \frac{1}{2}x^2}{x^4} \underset{\frac{0}{0}}{\overset{l'H}{=}} \lim_{x\to 0} \dfrac{-\sin x + x}{4x^3}$

$\underset{\frac{0}{0}}{\overset{l'H}{=}} \lim_{x\to 0} \dfrac{-\cos x + 1}{12x^2}$

$\underset{\frac{0}{0}}{\overset{l'H}{=}} \lim_{x\to 0} \dfrac{\sin x}{24x}$

$\underset{\frac{0}{0}}{\overset{l'H}{=}} \lim_{x\to 0} \dfrac{\cos x}{24} = \dfrac{\cos 0}{24} = \dfrac{1}{24}$

(b) $\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty} \dfrac{\sin(\pi/x)}{1/x}$

$\infty \cdot 0$

$\underset{\frac{0}{0}}{\overset{l'H}{=}} \lim_{x\to\infty} \dfrac{\cos(\pi/x)\,(-\pi/x^2)}{-1/x^2}$

$= \lim_{x\to\infty} -\pi \cos(\pi/x)$

$= -\pi \cos(0) = -\pi.$

(c) $\lim_{x\to 0}(1-2x)^{1/x} = \lim_{x\to 0} e^{\ln((1-2x)^{1/x})}$

$= \lim_{x\to 0} e^{\frac{1}{x}\ln(1-2x)}$

$= e^{\lim_{x\to 0} \frac{\ln(1-2x)}{x}} = e^{-2}$

$\lim_{x\to 0} \dfrac{\ln(1-2x)}{x} \underset{\frac{0}{0}}{\overset{l'H}{=}} \lim_{x\to 0} \dfrac{\frac{1}{1-2x}(-2)}{1} = \lim_{x\to 0} \dfrac{-2}{1-2x} = -2$

Note: I had intended to change this problem. As a result, this problem was not included in the total points for this test.

3. (20 points) Create a sketch of $g(t) = \frac{t^2-4}{t^2-2t}$ that shows the most important aspects of the graph.

For full credit be sure you address each of the eight guidelines for sketching the most important aspects of a function. (Domain, Intercepts, Symmetry, Asymptotes, Intervals of Increase or Decreases, Local Max and Local Min Values, Concavity and Points of Inflection, Sketch the Graph). If you do not use one of these steps, please indicate why.

The best way to look at this function is to simplify it!

$g(t) = \dfrac{t^2-4}{t^2-2t}$ is not defined when $t^2-2t = t(t-2) = 0$,

that is, when $t=0$ or $t=2$.

For all other values of t: $g(t) = \dfrac{t^2-4}{t^2-2t} = \dfrac{\cancel{(t-2)}(t+2)}{t\,\cancel{(t-2)}} = \dfrac{t+2}{t} = 1+\dfrac{2}{t}.$

$\lim\limits_{t\to 0^+} g(t) = \lim\limits_{t\to 0^+} 1+\dfrac{2}{t} = +\infty$

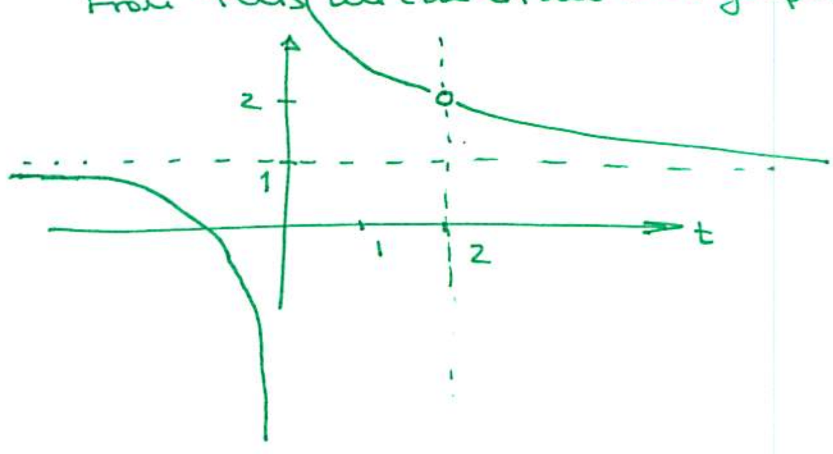
From this we can draw the graph.

$\lim\limits_{t\to 0^-} g(t) = \lim\limits_{t\to 0^-} 1+\dfrac{2}{t} = -\infty$

$\lim\limits_{t\to 2} g(t) = 1+\dfrac{2}{2} = 1+1 = 2.$

$\lim\limits_{t\to \infty} g(t) = \lim\limits_{t\to \infty} 1+\dfrac{2}{t} = 1+0 = 1$

$\lim\limits_{t\to -\infty} g(t) = \lim\limits_{t\to -\infty} 1+\dfrac{2}{t} = 1+0 = 1.$



If you don't see this, you can also use the more traditional approach.

$g'(t) = \dfrac{(t^2-2t)(2t)-(t^2-4)(2t-2)}{(t^2-2t)^2} = \dfrac{t\cancel{(t-2)}(2t)-\cancel{(t-2)}(t+2)2(t-1)}{t^2(t-2)^{\cancel{2}}} = \dfrac{2t^2-2(t^2+t-2)}{t^2(t-2)}$

$= \dfrac{-2\cancel{(t-2)}}{t^2\cancel{(t-2)}} = -\dfrac{2}{t^2}$ which is always negative.

$g''(t) = -2(-2t^{-3}) = \dfrac{4}{t^3}$ which is positive if $t>0$ and negative if $t<0$.

From the asymptotes and concavity, we can complete the graph as above.

4. (20 points) Create a sketch of $f(x) = x\sqrt{2-x^2}$ that shows the most important aspects of the graph.

For full credit be sure you address each of the eight guidelines for sketching the most important aspects of a function. (Domain, Intercepts, Symmetry, Asymptotes, Intervals of Increase or Decreases, Local Max and Local Min Values, Concavity and Points of Inflection, Sketch the Graph). If you do not use one of these steps, please indicate why.

NOTE: $f'(x) = -2\dfrac{x^2-1}{\sqrt{2-x^2}}$ and $f''(x) = \dfrac{2x(x^2-3)}{(2-x^2)^{3/2}}$.

Domain: need $2-x^2 \geq 0$ so $x^2 \leq 2$ or $-\sqrt{2} \leq x \leq \sqrt{2}$.

Intercepts: $f(0) = 0$.

$f(x) = 0 \iff x\sqrt{2-x^2} = 0 \iff x = 0$ or $2 - x^2 = 0$
$x^2 = 2$
$x = \pm 2$

x	$f(x)$
0	0 ← infl. pt
$-\sqrt{2}$	0
$\sqrt{2}$	0
-1	-1 ← local min
1	1 ← local max

Symmetry: $f(-x) = (-x)\sqrt{2-(-x)^2} = -x\sqrt{2-x^2} = -f(x)$

This is an odd function (symmetric about origin).

Asymptotes: $\lim_{x\to\pm\infty} f(x)$ do not make sense since domain ends at $\pm\sqrt{2}$.
no vertical asymptotes.

Increase / Decrease: $f'(x)$ dne. when $2-x^2=0$ i.e. $x = \pm\sqrt{2}$ (endpts. of domain).

$f'(x) = 0 \iff x^2 - 1 = 0 \iff x = \pm 1$.

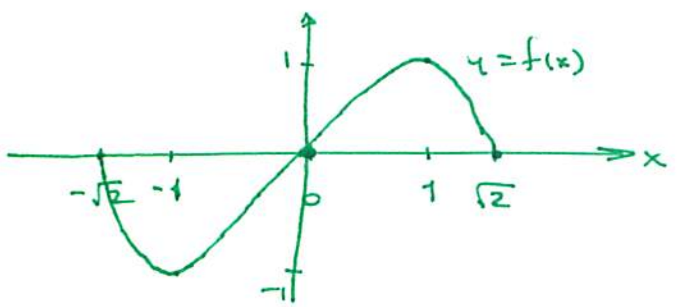
f': dne $-$ 0 $+$ 0 $-$ dne, at $-\sqrt{2}$, -1, 0, 1, $\sqrt{2}$ on the x axis; -1 ← local min, 1 ← local max.

Local Max/Min: From sign chart: max @ $x = 1$
min @ $x = -1$

f'': $+$ 0 $-$, at 0 ← infl. pt.

Concavity: $f''(x) = 0 \iff 2x(x^2-3) = 0 \iff x = 0$ or $x^2 = 3$
$x = \pm\sqrt{3}$ ← not in domain.

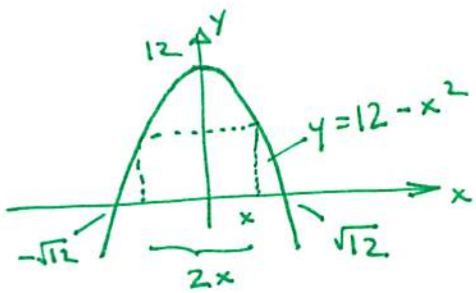
Sketch:



5. (16 points) Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 12 - x^2$.

(a) What function are you optimizing?

$$\max A = 2xy$$
$$= 2x(12 - x^2)$$
$$= 24x - 2x^3$$



(b) On what interval?

Because $x \geq 0$ and $y \geq 0$ are needed to make sense of the area,

$y = 0 \iff 12 - x^2 = 0$
$\iff 12 = x^2$
$\iff x = \pm\sqrt{12}$

The interval is $[0, \sqrt{12}]$.

(c) Answer the question.

$$\frac{dA}{dx} = 24 - 6x^2 = 0$$
$$6x^2 = 24$$
$$x^2 = 4$$
$$x = \pm 2$$

x	A
0	0
2	$48 - 16 = 32$
$\sqrt{12}$	0

Largest rectangle has sides $2x = 4$ & $y = 12 - x^2 = 12 - 4 = 8$, which has area $4 \cdot 8 = 32$.

6. (14 points) Suppose that f is a differentiable function with $f(2) = 12$ and $3 \le f'(x) \le 5$ for all values of x.

HINT: Use the Mean Value Theorem on the interval $[2, 8]$, twice.

(a) Explain how you know that $30 \le f(8)$.

By the MVT we know that for each x in $[2, 8]$.

$$\frac{f(x) - f(2)}{x - 2} = f'(c) \quad \text{for some } c \text{ between } 2 \text{ and } x$$

Because $f'(c) \geq 3$ this becomes (with $x = 8$):

$$\frac{f(8) - f(2)}{8 - 2} = f'(c) \geq 3.$$

$$\frac{f(8) - 12}{6} \geq 3$$

$$f(8) - 12 \geq 3 \cdot 6 = 18$$

$$f(8) \geq 18 + 12 = 30.$$

(b) How large can $f(8)$ be? That is, find a number M such that $f(8) \le M$.

In the same way as in (a), except that we use $f'(c) \le 5$:

$$\frac{f(8) - f(2)}{8 - 2} = f'(c) \le 5$$

$$\frac{f(8) - 12}{6} \le 5$$

$$f(8) - 12 \le 5 \cdot 6 = 30$$

$$f(8) \le 42 \quad (\text{so } M = 42).$$