MATH 141 (Section 11 & 12)
Prof. Meade

Exam 4 – Practice
November 19, 2007

University of South Carolina
Fall 2007

Name: ______________________
Section: 011 / 012 (circle one)

Instructions:

1. There are a total of 7 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	20	
2	15	
3	15	
4	15	
5	15	
6	10	
7	10	
Total	100	

Study Smart!

1. (20 points) Let $f(x) = xe^{-x/2}$. Find

   (a) the interval(s) on which f is increasing

   (b) the interval(s) on which f is decreasing

   (c) the open intervals on which f is concave up

   (d) the open interval(s) on which f is concave down

   (e) the x-coordinates of all inflection points

NOTE: Be sure to show your work and to label your answers clearly.

$$f'(x) = e^{-x/2} + x\left(-\tfrac{1}{2}\right)e^{-x/2} = \left(1 - \tfrac{x}{2}\right)e^{-x/2}$$

$$f''(x) = -\tfrac{1}{2}e^{-x/2} + \left(1-\tfrac{x}{2}\right)\left(-\tfrac{1}{2}\right)e^{-x/2} = \left(-1 + \tfrac{x}{4}\right)e^{-x/2}$$

$$f'(x) = 0 \iff \left(1 - \tfrac{x}{2}\right)\underbrace{e^{-x/2}}_{>0} = 0 \iff \left(1-\tfrac{x}{2}\right) = 0 \iff x = 2$$

$f'(x)$: $+$ on the left of 2, 0 at 2, $-$ on the right of 2

(a) f is increasing on $(-\infty, 2)$

(b) f is decreasing on $(2, \infty)$

$$f''(x) = 0 \iff \left(-1 + \tfrac{x}{4}\right)e^{-x/2} = 0 \iff -1 + \tfrac{x}{4} = 0 \iff x = 4$$

$f''(x)$: $-$ on the left of 4, 0 at 4, $+$ on the right of 4

(c) f is concave up on $(4, \infty)$

(d) f is concave down on $(-\infty, 4)$

(e) f has an inflection point when $x = 4$

$= (2-3x)(x+2)^{-1/3}$

2. (15 points) Let $f'(x) = \frac{2-3x}{\sqrt[3]{x+2}}$ be the first derivative of a continuous function f. Find all critical points of f and determine whether each is a relative maximum, relative minimum, or neither.

NOTE: Show enough work to justify your answers.

Critical points occur when $f'(x) = 0$ or $f'(x)$ dne.

$$f'(x) = 0 \iff \frac{2-3x}{\sqrt[3]{x+2}} = 0 \iff 2-3x = 0 \iff x = 2/3$$

$$f'(x) \text{ dne} \iff \sqrt[3]{x+2} = 0 \iff x+2 = 0 \iff x = -2.$$

Note that $\sqrt[3]{x+2} = (x+2)^{1/3}$ is defined for all x.)

There are 2 critical points: at $x = -2$ and at $x = 2/3$.

Using the First-Derivative Test:

$f'(x)$	$-$	dne	$+$	0	$-$
		-2		$2/3$	

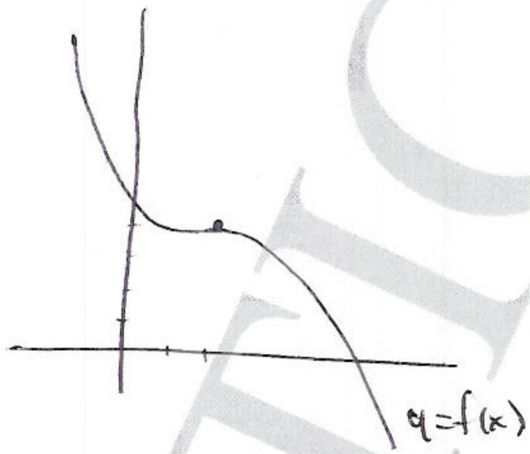
At $x = -2$ the function has neither a local max nor a local min.

At $x = 2/3$ the function has a local max

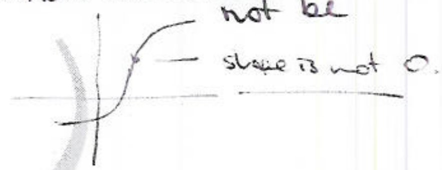
3. (15 points) Sketch the graph of a continuous curve $y = f(x)$ with the following properties: $f(2) = 4$, $f'(2) = 0$, $f''(x) > 0$ for $x < 2$, $f''(x) > 0$ for $x > 2$.

$f'(x)$ — 0 at 2

$f''(x)$ — $+$ for $x < 2$, $-$ for $x > 2$



Note: The function could ~~also be~~ not be increasing



4. (15 points) Find the absolute maximum and absolute minimum values of $f(x) = \sin(x) - \cos(x)$ on $[0, \pi]$.

$f'(x) = \cos(x) + \sin(x)$

$f'(x) = 0 \iff \cos(x) + \sin(x) = 0$

$\sin(x) = -\cos(x)$

$\tan(x) = -1 \implies x = \frac{-\pi}{4}$ or $x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4} \ldots$

x	$f(x)$
0	$0 - 1 = -1$
π	$0 - (-1) = 1$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2} \approx 1.4$

Absolute max is $f\left(\frac{3\pi}{4}\right) = \sqrt{2}$

Absolute min is $f(0) = -1$.

5. (15 points) Consider the following applied optimization problem:

> A closed rectangular container with a square base is to have a volume of 2000 cm^3. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimensions of the container of least cost.

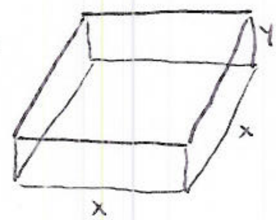
Find

(a) the function to be maximized or minimized (indicate which it is)

(b) the interval of possible values for the variable in this problem

Do not solve the optimization problem.

$$V = x^2 y = 2000 \implies y = \frac{2000}{x^2}$$



$$C = 2(x^2 + x^2) + 1(xy + xy + xy + xy)$$
$$= 4x^2 + 4xy$$
$$= 4x^2 + 4x\left(\frac{2000}{x^2}\right)$$
$$= 4x^2 + \frac{8000}{x}$$

(a) minimize $C(x) = 4x^2 + \frac{8000}{x}$

(b) for x in $(0, \infty)$.

6. (10 points) Determine if the hypotheses of the Mean-Value Theorem are satisfied for $f(x) = \frac{1}{x-1}$ on the interval $[2, 5]$. If they are, find all values of c in this interval that satisfy the conclusion of the theorem.

$$f(x) = \frac{1}{x-1} = (x-1)^{-1}$$

$$f'(x) = -(x-1)^{-2} = \frac{-1}{(x-1)^2}$$

both exist everywhere except $x = 1$.

f is continuous on $[2,5]$
f' exists on $(2,5)$.
} so the MVT applies.

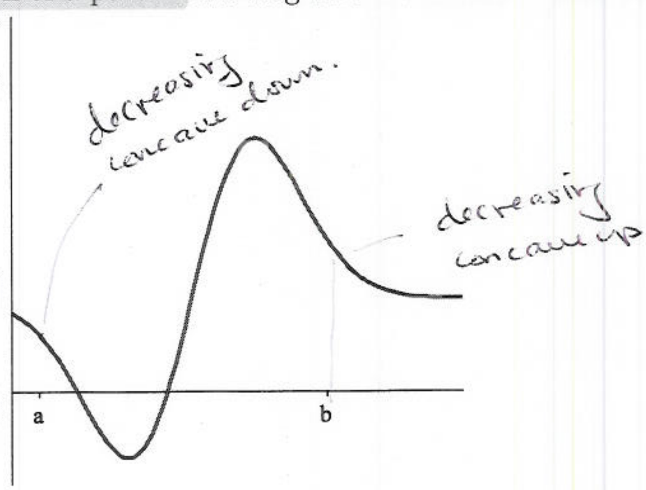
Find c in $(2,5)$ with $f'(c) = \frac{f(5)-f(2)}{5-2}$

$$-\frac{1}{(c-1)^2} = \frac{\frac{1}{5-1} - \frac{1}{2-1}}{5-2} = \frac{\frac{1}{4}-1}{3} = \frac{-3/4}{3} = -1/4$$

Solve
$$\frac{-1}{(c-1)^2} = \frac{-1}{4}$$
$$(c-1)^2 = 4$$
$$c-1 = \pm 2$$
$$c = 1 \pm 2 = 3, -1$$

The conclusion of the MVT is satisfied when $c = 3$.

7. (10 points) The position function of a particle moving on a horizontal x-axis is shown below.



(a) Is the particle moving left or right at time a? decreasing ⟹ moving left

(b) Is the acceleration positive, negative, or zero at time a? concave down ⟹ negative

(c) Is the particle speeding up or slowing down at time a? decr. & c. down ⟹ speeding up.

(d) Is the particle speeding up or slowing down at time b? decr & c. up ⟹ slowing down.