MATH 141 (Section 1 & 2)
Prof. Meade

University of South Carolina
Fall 2004

Exam 4
November 23, 2004

Name: Key
Section: 001 002 (circle one)

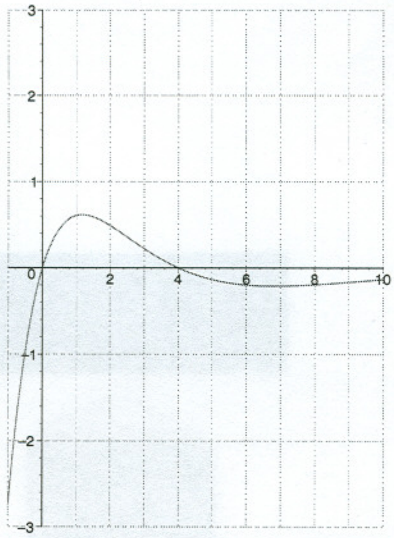
Instructions:

1. There are a total of 6 problems (including the Extra Credit problem) on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	20	
2	16	
3	10	
4	9	
5	25	
6	20	
Extra Credit	5	
Total	100	

Happy Thanksgiving!

1. (20 points) Use the following graph of $y = f'(x)$ to answer the following questions about the unknown function f.



Hint: *The numbers appearing in your answers must be taken from the following list:*
-1.00, 0.00, 1.17, 4.00, 6.83, 10.00

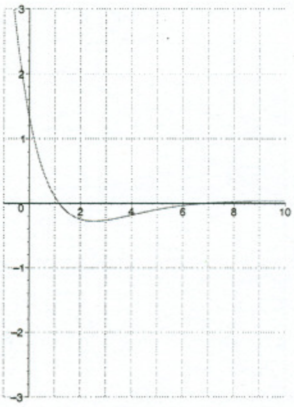
(a) On what interval(s) is f increasing? $f'(x) > 0$ for $0 < x < 4$

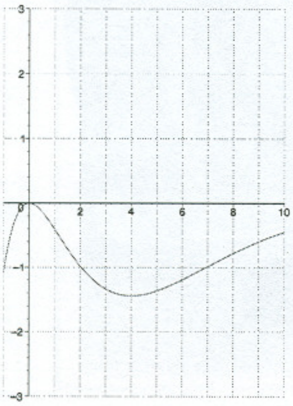
(b) Give the x-coordinate(s) of all points where f has a relative maximum.
rel. max. when $f'(x)$ changes from + to − (f changes from incr. to decr.)
this occurs only when $x = 4$.

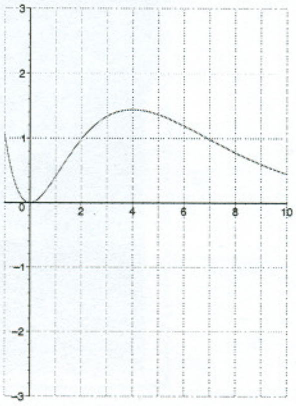
(c) On what interval(s) is f concave down? f' decreasing for $1.17 < x < 6.83$.

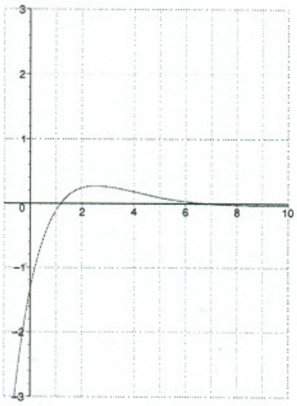
(d) Which of the following graphs is the graph of $y = f(x)$? [Circle one: (i) (ii) (iii) (iv)]

(e) Which of the following graphs is the graph of $y = f''(x)$? [Circle one: (i) (ii) (iii) (iv)]


(i)


(ii)


(iii)


(iv)

2. (16 points) Let $s(t) = \dfrac{100}{t^2+12}$ be the position function of a particle moving along a coordinate line, where s is in feet and t is in seconds ($t \geq 0$).

(a) Show that the general formula for the velocity is $v(t) = \dfrac{-200t}{(t^2+12)^2}$.

$$v(t) = s'(t) = \frac{d}{dt}\, 100\,(t^2+12)^{-1}$$
$$= 100\,(-1)\,(t^2+12)^{-2}\,(2t)$$
$$= -200t\,(t^2+12)^{-2}$$
$$= \frac{-200t}{(t^2+12)^2}$$

(b) The general formula for the acceleration is $a(t) = \dfrac{600(t^2-4)}{(t^2+12)^3}$. Find the maximum speed of the particle. At what time does the particle attain its maximum speed?

max. speed occurs at a critical number for the speed.

critical numbers for the velocity (& speed) are solutions to $v'(t)=0$.

because $v'(t) = a(t) = 0$ for $t^2 = 4$, i.e., $t = \pm 2$, we look at

① $|v(2)| = \left|\frac{-400}{16^2}\right| = \frac{400}{256}$ ← maximum speed is $\frac{400}{256}$ ft/sec at $t=2$sec

② $|v(0)| = \left|\frac{0}{12^2}\right| = 0$

③ $\lim\limits_{t\to\infty} |v(t)| = \lim\limits_{t\to\infty} \frac{200t}{(t^2+12)^2} = 0.$ } to be sure the max speed does not occur at an "endpoint".

(c) Find the position of the particle when it has its maximum speed.

$s(2) = \frac{100}{16}$ ft.

(d) Find the direction of motion when it has its maximum speed.

$v(2) = \frac{-400}{16^2} = -\frac{400}{256}$ so the particle is moving back towards the origin (to the left).

3. (10 points) Find the absolute maximum and minimum values of $f(x) = 2x^3 - 3x^2 - 12x$ on the interval $-2 \le x \le 1$. **Note:** $f(-2) = -4$ and $f(1) = -13$.

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x^2 - x - 2)$$
$$= 6(x-2)(x+1)$$

critical numbers: $x = 2$ (not in interval)
$x = -1$

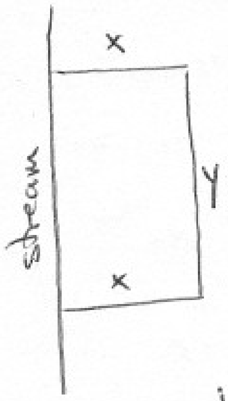
x	$f(x)$	
-2	-4	
1	-13	← abs. min is -13 when $x = 1$
-1	7	← abs. max is 7 when $x = -1$.

4. (9 points) Consider the problem:

A rectangular field is to be bounded by a fence on three sides and by a straight stream on the fourth side. Find the dimensions of the field with maximum area that can be enclosed with 1000 feet of fence.

Formulate the problem as a max/min problem on an appropriate interval.

Do not solve the problem!



total length of fence needed: $x + y + x = 1000$
$2x + y = 1000$

area enclosed: $A = xy$.

to eliminate a variable: $2x + y = 1000 \iff y = 1000 - 2x$.

$$\boxed{\text{maximize } A(x) = x(1000 - 2x) \quad 0 \le x \le 500}$$

$x \ge 0$
$y \ge 0 \iff 1000 - 2x \ge 0$
$1000 \ge 2x$
$500 \ge x$

if you solve for x: $x = 500 - \frac{y}{2}$

maximize $A(y) = (500 - \frac{y}{2})y$
$0 \le y \le 1000$

5. (25 points) Evaluate each of the following indefinite integrals.

(a) $\int x + x^5 + x^{-2} - 4x^{1/3}\, dx$ $= \frac{1}{2}x^2 + \frac{1}{6}x^6 + (\frac{1}{-1})x^{-1} - 4\frac{1}{4/3}x^{4/3} + C$

$= \frac{1}{2}x^2 + \frac{1}{6}x^6 - x^{-1} - 3x^{4/3} + C$

(b) $\int \left(\frac{1}{2t} + 3\sqrt{t} + 2\cos t - \sqrt{2}e^t\right)\, dt$ $= \frac{1}{2}\int \frac{1}{t}\,dt + 3\int t^{1/2}dt + 2\int \cos t\,dt$

$- \sqrt{2}\int e^t dt$

$= \frac{1}{2}\ln|t| + 3\frac{1}{3/2}t^{3/2} + 2\sin t - \sqrt{2}e^t + C$

$= \frac{1}{2}\ln|t| + 2t^{3/2} + 2\sin t - \sqrt{2}e^t + C$

(c) $\int 2x(x^2-2)^{23}\, dx$ $= \int u^{23}\,du$

$u = x^2 - 2$

$du = 2x\,dx$

$= \frac{1}{24}u^{24} + C$

$= \frac{1}{24}(x^2-2)^{24} + C$

(d) $\int \frac{1}{x\sqrt{\ln x}}\, dx$ $= \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}du = \frac{1}{1/2}u^{1/2} + C$

$= 2\sqrt{\ln x} + C.$

$u = \ln x$

$du = \frac{1}{x}dx$

(e) $\int \frac{\sin x}{1+\cos^2 x}\, dx$ $= -\int \frac{-\sin x}{1+\cos^2 x}\,dx = -\int \frac{1}{1+u^2}\,du$

$u = \cos x$

$= -\arctan u + C$

$du = -\sin x\,dx$

$= -\arctan(\cos x) + C.$

6. (20 points)

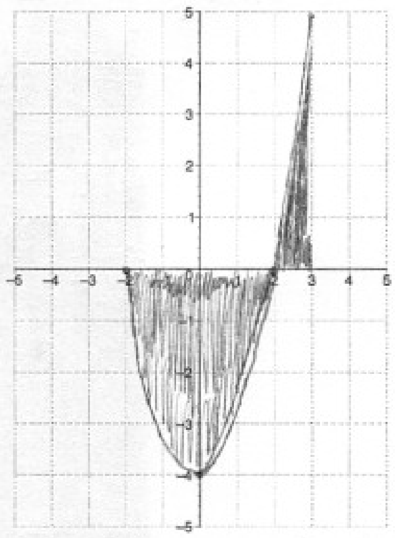
(a) Express the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \frac{\sin^2(x_k^*)}{1+x_k^*} \Delta x_k = \int_0^{\pi} \frac{\sin^2 x}{1+x}\, dx$$

as a definite integral with $a = 0$ as the lower limit of integration and $b = \pi$ as the upper limit of integration. *Do not evaluate the integral!*

(b) Use the axes provided to sketch the region whose net signed area is represented by the definite integral

$$\int_{-2}^{3} x^2 - 4\, dx.$$



Based on your sketch, do you expect this definite integral to have a positive or negative value? negative. *Do not evaluate the integral!*

(c) Use properties of the definite integral and appropriate geometric formulas to evaluate

$$\int_{-1}^{1} 3 + 2\sqrt{1-x^2}\, dx. = 6+\pi.$$

$\int_{-1}^{1} 3\, dx =$ area of rectangle w/ height 3 & width 2 $= 3(2) = 6.$

$\int_{-1}^{1} 2\sqrt{1-x^2}\, dx = 2$ (area of semicircle w/ radius 1) $= 2\left(\frac{1}{2}\pi(1)^2\right) = 2\frac{\pi}{2} = \pi.$

(d) Suppose $\int_0^2 f(x)\, dx = -2$, $\int_2^3 f(x)\, dx = 1$, and $\int_0^3 g(x)\, dx = 3$. Find $\int_0^3 f(x)+2g(x)\, dx$.

$$\int_0^3 f(x)+2g(x)dx = \int_0^2 f(x)dx + \int_2^3 f(x)dx + 2\int_0^3 g(x)dx = -2+1+2(3) = 5.$$

Extra Credit (5 points)State Rolle's Theorem.

If 1) f is differentiable on (a,b)
2) f is continuous on $[a,b]$
3) $f(a) = f(b) = 0$

then there is at least one number c in (a,b) for which
$f'(c) = 0.$