MATH 141 (Section 1 & 2)
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University of South Carolina
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Exam 2
October 6, 2004

Name: Key
Section: 001 002 (circle one)

Instructions:

1. There are a total of 7 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	36	
2	12	
3	8	
4	16	
5	10	
6	10	
7	8	
Total	100	

1. (36 points) Find each requested derivative.

(a) [8 points] Find $f'(x)$ when $f(x) = x^{10} + 3x^{2/3} + \frac{4}{\sqrt{x}}$ $= x^{10} + 3x^{2/3} + 4x^{-1/2}$

$$f'(x) = 10x^9 + 3\left(\tfrac{2}{3}\right)x^{-1/3} + 4\left(-\tfrac{1}{2}\right)x^{-3/2}$$
$$= 10x^9 + 2x^{-1/3} - 2x^{-3/2}$$

(b) [8 points] Find $\frac{dy}{dx}$ when $y = x\cos(x^2)$

$$\frac{dy}{dx} = (1)\cos(x^2) + x(-\sin(x^2)\cdot 2x)$$
$$= \cos(x^2) - 2x^2\sin(x^2)$$

(c) [8 points] Find y' when $y = \tan(t^2 + 4t + 3)$

$$y' = \sec^2(t^2+4t+3)\cdot(2t+4)$$
$$= (2t+4)\sec^2(t^2+4t+3)$$

(d) [12 points] Find $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ when $x = \frac{t-1}{t+1}$

NOTE: *Simplify* $\frac{dx}{dt}$ *before finding* $\frac{d^2x}{dt^2}$.

$$\frac{dx}{dt} = \frac{(t+1)(1) - (t-1)(1)}{(t+1)^2} = \frac{t+1-t+1}{(t+1)^2} = \frac{2}{(t+1)^2} = 2(t+1)^{-2}$$

$$\frac{d^2x}{dt^2} = 2(-2)(t+1)^{-3}(1) = -4(t+1)^{-3}$$

2. (12 points) Let $f(x) = \frac{x}{x^2+9}$.

(a) Find $f'(x)$.

$$f'(x) = \frac{(x^2+9)(1) - x(2x)}{(x^2+9)^2} = \frac{x^2+9-2x^2}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2}.$$

(b) Find all points on the graph of $y = f(x)$ where the tangent line to the graph is horizontal.

The tangent line to the graph of $y = f(x)$ is horizontal at all points where $f'(x) = 0$. With $f'(x) = \frac{9-x^2}{(x^2+9)^2}$ we see that $f'(x) = 0$ at all points where $9 - x^2 = 0$, i.e. $x^2 = 9$

$x = \pm 3.$

The points on the graph with horizontal tangent lines are

$(3, \frac{3}{18})$ and $(-3, \frac{-3}{18})$

or $(3, 1/6)$ $(-3, -1/6)$.

3. (8 points) Let f be a differentiable function with $f'(x) = \sqrt{3x+7}$, $g(x) = x^2 - 1$, and $F(x) = f(g(x))$. Find $F'(x)$.

NOTE: *Be sure to simplify your answer.*

By the Chain Rule:

$$F'(x) = f'(g(x))\, g'(x)$$

$$= \sqrt{3g(x)+7} \cdot g'(x)$$

$$= \sqrt{3(x^2-1)+7} \cdot (2x)$$

$$= 2x\sqrt{3x^2+4}.$$

4. (16 points) A Folium of Descartes is defined implicitly by the equation $x^3 + y^3 = 3xy$.

(a) Use implicit differentiation to find $\frac{dy}{dx}$.

$$x^3 + y(x)^3 = 3x\,y(x).$$

$$\frac{d}{dx}\left(x^3 + y(x)^3\right) = \frac{d}{dx}\left(3x\,y(x)\right)$$

$$3x^2 + 3y(x)^2\frac{dy}{dx} = 3\left(1\cdot y(x) + x\frac{dy}{dx}\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

$$(3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}$$

(b) Find an equation for the tangent line to the Folium of Descartes at the point $(\frac{3}{2}, \frac{3}{2})$.

First, note that $\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^3 = \frac{27}{8} + \frac{27}{8} = \frac{27}{4}$.

and $3\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^2 = \frac{27}{4}$

So the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ is on this curve.

At this point, the slope of the tangent line is $m = \frac{dy}{dx} = \frac{\frac{3}{2} - \left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2 - \frac{3}{2}} = \frac{\frac{3}{2} - \frac{9}{4}}{\frac{9}{4} - \frac{3}{2}} = \frac{-\frac{3}{4}}{\frac{3}{4}} =$

The tangent line is $y = -1\left(x - \frac{3}{2}\right) + \frac{3}{2} = -x + \frac{3}{2} + \frac{3}{2} = -x + 3.$

5. (10 points) Define

$$f(x) = \begin{cases} kx^2 & x \le 2 \\ kx + L & 2 < x < 4 \\ 2 + \frac{L}{x} & x \ge 4 \end{cases}.$$

Find, and solve, the equations that k and L must satisfy so that f is continuous for all real numbers.

To have continuity at $x=2$: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} kx^2 = 4k$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} kx+L = 2k+L$

$f(2) = 4k.$

To have $\lim_{x\to 2} f(x) = f(2)$ we must have $2k+L = 4k$, or $\underline{L = 2k}$.

To have continuity at $x=4$: $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} kx+L = 4k+L$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} 2+\frac{L}{x} = 2+\frac{L}{4}.$

$f(4) = 2+\frac{L}{4}.$

To have $\lim_{x\to 4} f(x) = f(4)$ we must have $4k+L = 2+\frac{L}{4}$, or $\underline{4k+\frac{3}{4}L = 2}$.

6. (10 points) Evaluate the following limits. If a limit does not exist, explain why.

(a) $\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \lim_{x\to 0} \frac{5x}{\sin 5x} \cdot \lim_{x\to 0} \frac{3x}{5x}$

$= 1 \cdot 1 \cdot \frac{3}{5}$

$= \frac{3}{5}.$

To solve these equations:

$L = 2k \Rightarrow 2 = 4k + \frac{3}{4}L = 4k + \frac{3}{4}(2k)$

$= 4k + \frac{3}{2}k = \frac{11}{2}k.$

$\Rightarrow \boxed{k = \frac{4}{11} \text{ and } L = 2k = \frac{8}{11}}$

(b) $\lim_{\Delta x\to 0} \frac{\sqrt{1+\Delta x} - 1}{\Delta x}$

NOTE: *Recognize this limit as a derivative and evaluate the derivative.*

This looks like a difference quotient $\lim_{\Delta x\to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$

with $f(x) = \sqrt{x}$ and $a = 1$. Thus, $\lim_{\Delta x\to 0} \frac{\sqrt{1+\Delta x} - 1}{\Delta x} = \frac{d}{dx}\sqrt{x}\Big|_{x=1}$

$= \frac{1}{2\sqrt{x}}\Big|_{x=1}$

$= \frac{1}{2}.$

(or $f(x) = \sqrt{1+x}$ and $a = 0$)
in which case we need to compute:
$\frac{d}{dx}(1+x)^{1/2}\Big|_{x=0} = \frac{1}{2}(1+x)^{-1/2}(1)\Big|_{x=0} = \frac{1}{2}(1)^{-1/2} = \frac{1}{2}$)

7. (8 points) Let $f(x) = \dfrac{1}{x+1}$. Use the definition of the derivative to find $f'(2)$.

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2} \frac{\frac{1}{x+1} - \frac{1}{3}}{x-2}$$

$$= \lim_{x \to 2} \frac{\frac{3}{3(x+1)} - \frac{x+1}{3(x+1)}}{x-2}$$

$$= \lim_{x \to 2} \frac{3 - (x+1)}{3(x+1)(x-2)}$$

$$= \lim_{x \to 2} \frac{2 - x}{3(x+1)(x-2)}$$

$$= \lim_{x \to 2} \frac{-1}{3(x+1)}$$

$$= \frac{-1}{3(2+1)}$$

$$= -\frac{1}{9}.$$

Check: $f'(x) = (-1)(x+1)^{-2}(1)$

$= -(x+1)^{-2}$

So $f'(2) = -(3)^{-2} = -\frac{1}{9}.$