Chapter 9

Polar Coordinates and Plane Curves

This chapter presents further applications of the derivative and integral. Section 9.1 describes polar coordinates. Section 9.2 shows how to compute the area of a flat region that has a convenient description in polar coordinates. Section 9.3 introduces a method of describing a curve that is especially useful in the study of motion.

The speed of an object moving along a curved path is developed in Section 9.4. It also shows how to express the length of a curve as a definite integral. The area of a surface of revolution as a definite integral is introduced in Section 9.5. The sphere is an instance of such a surface.

Section 9.6 shows how the derivative and second derivative provide tools for measuring how curvy a curve is at each of its points. This measure, called "curvature," will be needed in Chapter 15 in the study of motion along a curve.

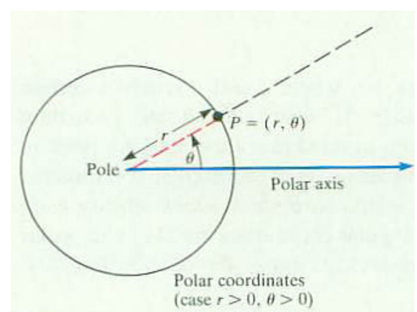
9.1 Polar Coordinates

Rectangular coordinates provide only one of the ways to describe points in the plane by pairs of numbers. This section describes another coordinate system called "polar coordinates."

Polar Coordinates

The rectangular coordinates x and y describe a point P in the plane as the intersection of two perpendicular lines. Polar coordinates describe a point P as the intersection of a circle and a ray from the center of that circle. They are defined as follows.

(a)



(b)

Figure 9.1.1:

When we say "The storm is 10 miles northeast," we are using polar coordinates: $r = 10$ and $\theta = \pi/4$.

Select a point in the plane and a ray emanating from this point. The point is called the **pole**, and the ray the **polar axis**. (See Figure 9.1.1(a).) Measure positive angles θ counterclockwise from the polar axis and negative angles clockwise. Now let r be a number. To plot the point P that corresponds to the pair of numbers r and θ, proceed as follows:

- If r is positive, P is the intersection of the circle of radius r whose center is at the pole and the ray of angle θ from the pole. (See Figure 9.1.1(b).)
- If r is 0, P is the pole, no matter what θ is.
- If r is negative, P is at a distance $|r|$ from the pole on the ray directly opposite the ray of angle θ, that is, on the ray of angle $\theta + \pi$.

In each case P is denoted (r, θ), and the pair r and θ are called the **polar coordinates** of P. The point (r, θ) is on the circle of radius $|r|$ whose center is the pole. The pole is the midpoint of the points (r, θ) and $(-r, \theta)$. Notice that the point $(-r, \theta + \pi)$ is the same as the point (r, θ). Moreover, changing the angle by 2π does not change the point; that is, $(r, \theta) = (r, \theta + 2\pi) = (r, \theta + 4\pi) = \cdots = (r, \theta + 2k\pi)$ for any integer k (positive or negative).

EXAMPLE 1 Plot the points $(3, \pi/4)$, $(2, -\pi/6)$, $(-3, \pi/3)$ in polar coordinates. See Figure 9.1.2.
SOLUTION

- To plot $(3, \pi/4)$, go out a distance 3 on the ray of angle $\pi/4$ (shown in Figure 9.1.2).
- To plot $(2, -\pi/6)$, go out a distance 2 on the ray of angle $-\pi/6$.
- To plot $(-3, \pi/3)$, draw the ray of angle $\pi/3$, and then go a distance 3 in the *opposite* direction from the pole.

◇

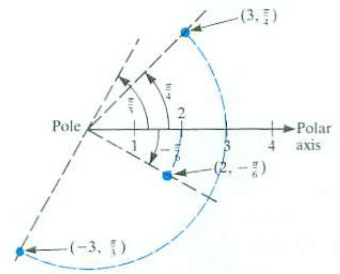


Figure 9.1.2:

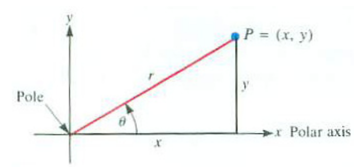


Figure 9.1.3:

The relation between polar and rectangular coordinates.

It is customary to have the polar axis coincide with the positive x-axis as in Figure 9.1.3. In that case, inspection of the diagram shows the relation between the rectangular coordinates (x, y) and the polar coordinates of the point P:

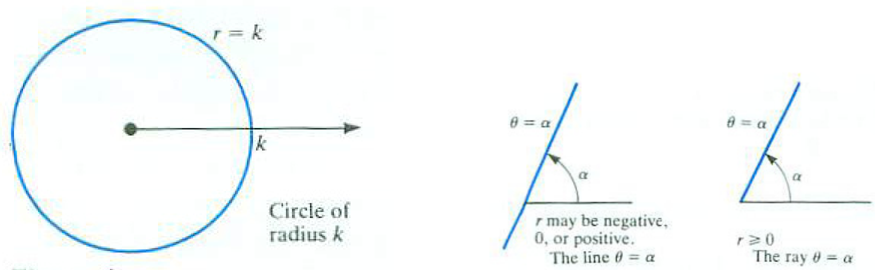
$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$
$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

These equations hold even if r is negative. If r is positive, then $r = \sqrt{x^2 + y^2}$. Furthermore, if $-\pi/2 < \theta < \pi/2$, then $\theta = \arctan(y/x)$.

Graphing $r = f(\theta)$

Just as we may graph the set of points (x, y), where x and y satisfy a certain equation, we may graph the set of points (r, θ), where r and θ satisfy a certain equation. Keep in mind that although each point in the plane is specified by a unique ordered pair (x, y) in rectangular coordinates, there are *many ordered pairs* (r, θ) *in polar coordinates that specify each point.* For instance, the point whose rectangular coordinates are $(1, 1)$ has polar coordinates $(\sqrt{2}, \pi/4)$ or $(\sqrt{2}, \pi/4 + 2\pi)$ or $(\sqrt{2}, \pi/4 + 4\pi)$ or $(-\sqrt{2}, \pi/4 + \pi)$ and so on.

The simplest equation in polar coordinates has the form $r = k$, where k is a positive constant. Its graph is the circle of radius k, centered at the pole. (See Figure 9.1.4(a).) The graph of $\theta = \alpha$, where α is a constant, is the line of inclination α. If we restrict r to be nonnegative, then $\theta = \alpha$ describes the ray ("half-line") of angle α. (See Figure 9.1.4(b).)



(a) (b)

Figure 9.1.4:

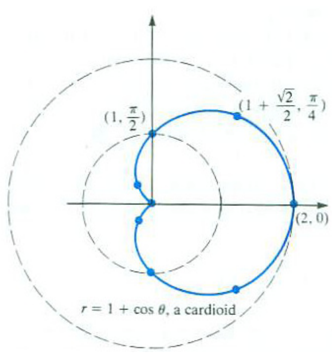


Figure 9.1.5: A cardioid is not shaped like a real heart, only like the conventional image of a heart.

EXAMPLE 2 Graph $r = 1 + \cos\theta$.
SOLUTION Begin by making a table: Since $\cos(\theta)$ has period 2π, we con-

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	2	$1+\frac{\sqrt{2}}{2}$ ≈ 1.7	1	$1-\frac{\sqrt{2}}{2}$ ≈ 0.3	0	$1-\frac{\sqrt{2}}{2}$ ≈ 0.3	1	$1+\frac{\sqrt{2}}{2}$ ≈ 1.7	2

Table 9.1.1:

sider only θ in $[0, 2\pi]$.

As θ goes from 0 to π, r decreases; as θ goes from π to 2π, r increases. The last point is the same as the first. The graph begins to repeat itself. This heart-shaped curve, shown in Figure 9.1.5, is called a **cardioid**. ◇

Spirals turn out to be quite easy to describe in polar coordinates. This is illustrated by the graph of $r = 2\theta$ in the next example.

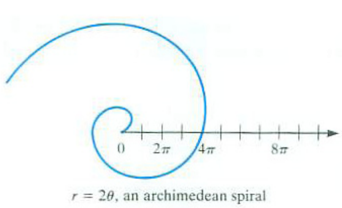


Figure 9.1.6:

EXAMPLE 3 Graph $r = 2\theta$ for $\theta \geq 0$.
SOLUTION First make a table:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	$\cdots$
r	0	π	2π	3π	4π	5π	$\cdots$

Increasing θ by 2π does *not* produce the same value of r. As θ increaes, r increases. The graph for $\theta \geq 0$ is an endless sprial, going infinitely often around the pole. It is indicated in Figure 9.1.6. ◇

If a is a nonzero constant, the graph of $r = a\theta$ is called an **Archimedean spiral** for a good reason: Archimedes was the first person to study the curve, finding the area within it up to any angle and also its tangent lines. The spiral with $a = 2$ is sketched in Example 3.

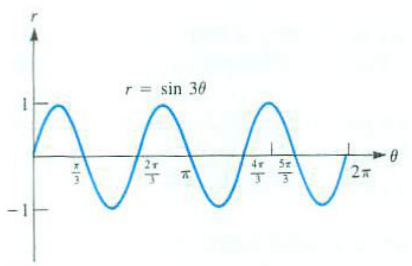
Polar coordinates are also convenient for describing loops arranged like the petals of a flower, as Example 4 shows.

EXAMPLE 4 Graph $r = \sin(3\theta)$.
SOLUTION Note that $\sin(3\theta)$ stays in the range -1 to 1. For instance, when $3\theta = \pi/2$, $\sin(3\theta) = \sin(\pi/2) = 1$. That tells us that when $\theta = \pi/6$, $r = \sin(3\theta) = \sin(3(\pi/6)) = \sin(\pi/2) = 1$. This case suggest that we calculate r at integer multiples of $\pi/6$, as in Table 9.1.2: The variation of r as a function

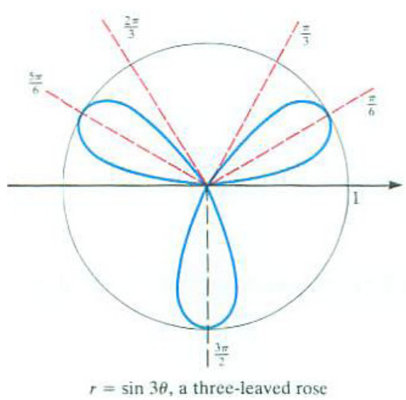
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
3θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{9\pi}{2}$	6π
$r = \sin(3\theta)$	0	1	0	-1	0	1	0	1	0

Table 9.1.2:

of θ is shown in Figure 9.1.7(a). Because $\sin(\theta)$ has period 2π, $\sin(3\theta)$ has period $2\pi/3$.



(a)

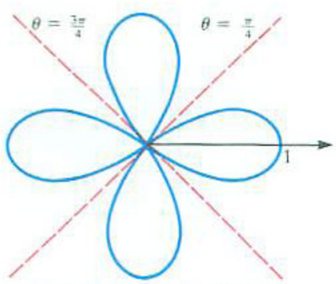


(b)

Figure 9.1.7:

As θ increases from 0 up to $\pi/3$, 3θ increases from 0 up to π. Thus r, which is $\sin(3\theta)$, starts at 0 (for $\theta = 0$) up to 1 (for $\theta = \pi/6$) and then back to 0 (for $\theta = \pi/3$). This gives one of the three loops that make up the graph of $r = \sin(3\theta)$. For θ in $[\pi/3, 2\pi/3]$, $r = \sin(3\theta)$ is negative (or 0). This yields the lower loop in Figure 9.1.7(b). For θ in $[2\pi/3, \pi]$, r is again positive, and we obtain the upper left loop. Further choices of θ lead only to repetition of the loops already shown. ◇

The graph of $r = \sin(n\theta)$ or $r = \cos(n\theta)$ has n loops when n is an odd integer and $2n$ loops when n is an even integer. The next example illustrates the case when n is even.



$r = \cos 2\theta$, a four-leaved rose

Figure 9.1.8:

EXAMPLE 5 Graph the four-leaved rose, $r = \cos(2\theta)$.
SOLUTION To isolate one loop, find the two smallest nonnegative values of θ for which $\cos(2\theta) = 0$. These values are the θ that satisfy $2\theta = \pi/2$ and $2\theta = 3\pi/2$; thus $\theta = \pi/4$ and $\theta = 3\pi/4$. One leaf is described by letting θ go from $\pi/4$ to $3\pi/4$. For θ in $[\pi/4, 3\pi/4]$, 2θ is in $[\pi/2, 3\pi/2]$. Since 2θ is then a second- or third-quadrant angle, $r = \cos(2\theta)$ is *negative* or 0. In particular, when $\theta = \pi/2$, $\cos(2\theta)$ reaches its smallest value, -1. This loop is the bottom one in Figure 9.1.8. The other loops are obtained similarly. Of course, we could also sketch the graph by making a table of values. ◇

EXAMPLE 6 Transform the equation $y = 2$, which describes a horizontal straight line, into polar coordinates.
SOLUTION Since $y = r\sin\theta$, $r\sin\theta = 2$, or

$$r = \frac{2}{\sin(\theta)} = 2\csc(\theta).$$

This is more complicated than the Cartesian version of this equation, but is still sometimes useful. ◇

EXAMPLE 7 Transform the equation $r = 2\cos(\theta)$ into rectangular coordinates and graph it.
SOLUTION Since $r^2 = x^2 + y^2$ and $r\cos\theta = x$, first multiply the equation $r = 2\cos\theta$ by r, obtaining

$$r^2 = 2r\cos(\theta)$$

Hence

$$x^2 + y^2 = 2x.$$

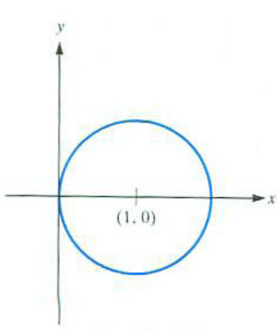
To graph this curve, rewrite the equation as

$$x^2 - 2x + y^2 = 0$$

and complete the square, obtaining

$$(x-1)^2 + y^2 = 1.$$

The graph is a circle of radius 1 and center at $(1, 0)$ in rectangular coordinates. It is graphed in Figure 9.1.9. ◇



$r = 2\cos\theta$, a circle

Figure 9.1.9:

Caution: The step in Example 7 where we multiply by r deserves some attention. If $r = 2\cos(\theta)$, then certainly $r^2 = 2r\cos(\theta)$. However, if $r^2 = 2r\cos(\theta)$, it does not follow that $r = 2\cos(\theta)$. We can "cancel the r" only when r is not 0. If $r = 0$, it is true that $r^2 = 2r\cos(\theta)$, but it not necessarily true that $r = 2\cos(\theta)$. Since $r = 0$ satisfies the equation $r^2 = 2r\cos\theta$, the pole is on the curve $r^2 = 2r\cos\theta$. Luckily, it is also on the original curve $r = 2\cos(\theta)$, since $\theta = \pi/2$ makes $r = 0$. Hence the graphs of $r^2 = 2r\cos(\theta)$ and $r = 2\cos(\theta)$ are the same.

However, as you may check, the graphs of $r = 2 + \cos(\theta)$ and $r^2 = r(2 + \cos(\theta))$ are *not* the same. The origin lies on the second curve, but not on the first.

The Intersection of Two Curves

Finding the intersection of two curves in polar coordinates is complicated by the fact that a given point has many descriptions in polar coordinates. Example 8 illustrates how to find the intersection.

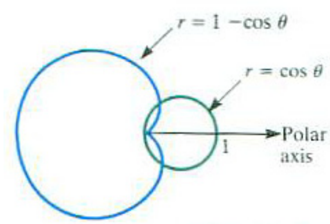


Figure 9.1.10:

EXAMPLE 8 Find the intersection of the curve $r = 1 - \cos(\theta)$ and the circle $r = \cos(\theta)$.
SOLUTION First graph the curves. The curve $r = \cos(\theta)$ is a circle half the size of the one in Example 7. Both curves are shown in Figure 9.1.10. (The curve $r = 1 - \cos(\theta)$ is a cardioid, being congruent to $r = 1 + \cos(\theta)$.) It appears that there are three points of intersection.

A point of intersection is produced when one value of θ yields the same value of r in both equations, we would have

$$1 - \cos(\theta) = \cos(\theta).$$

Hence $\cos(\theta) = \frac{1}{2}$. Thus $\theta = \pi/3$ or $\theta = -\pi/3$ (or any angle differing from these by $2n\pi$, n an integer). This gives two of the three points, but it fails to give the origin. Why?

How does the origin get to be on the circle $r = \cos(\theta)$? Because, when $\theta = \pi/2$, $r = 0$. How does it get to be on the cardioid $r = 1 - \cos(\theta)$? Because, when $\theta = 0$, $r = 0$. The origin lies on both curves, but we would not learn this by simply equating $1 - \cos(\theta)$ and $\cos(\theta)$. ◇

When checking for the intersection of two curves, $r = f(\theta)$ and $r = g(\theta)$ in polar coordinates, examine the origin separately. The curves may also interect at other points not obtainable by setting $f(\theta) = g(\theta)$. This possibility is due to the fact the point (r, θ) is the same as the points $(r, \theta + 2n\pi)$ and $(-r, \theta + (2n + 1)\pi)$ for any integer n. The safest procedure is to graph the two curves first, identify the intersections in the graph, and then see why the curves intersect there.

Summary

We introduced polar coordinates and showed how to graph curves given in the form $r = f(\theta)$. Some of the more common polar curves are listed below.

Equation	Curve
$r = a$, $a > 0$	circle of radius a, center at pole
$r = 1 + \cos(\theta)$	cardioid
$r = a\theta$, $a > 0$	Archimedean spiral (traced clockwise)
$r = \sin(n\theta)$, n odd	n-leafed rose (one loop symmetric about $\theta = \pi/n$)
$r = \sin(n\theta)$, n even	$2n$-leafed rose
$r = \cos(n\theta)$, n odd	n-leafed rose (one loop symmetric about $\theta = 0$)
$r = \cos(n\theta)$, n even	$2n$-leafed rose
$r = a\csc(\theta)$	the line $y = a$
$r = a\sec(\theta)$	the line $x = a$
$r = a\cos(\theta)$, $a > 0$	circle of radius $a/2$ through pole and $(a/2, 0)$
$r = a\sin(\theta)$, $a > 0$	circle of radius $a/2$ through pole and $(0, a/2)$

Table 9.1.3:

To find the intersection of two curves in polar coordinates, first graph them.

EXERCISES for Section 9.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] Plot the points whose polar coordinates are

(a) $(1, \pi/6)$

(b) $(2, \pi/3)$

(c) $(2, -\pi/3)$

(d) $(-2, \pi/3)$

(e) $(2, 7\pi/3)$

(f) $(0, \pi/4)$

2.[R] Find the rectangular coordinates of the points in Exercise 1.

3.[R] Give at least three pairs of polar coordinates (r, θ) for the point $(3, \pi/4)$,

(a) with $r > 0$,

(b) with $r < 0$.

4.[R] Find polar coordinates (r, θ) with $0 \le \theta < 2\pi$ and r positive, for the points whose rectangular coordinates are

(a) $(\sqrt{2}, \sqrt{2})$

(b) $(-1, \sqrt{3})$

(c) $(-5, 0)$

(d) $(-\sqrt{2}, -\sqrt{2})$

(e) $(0, -3)$

(f) $(1, 1)$

In Exercises 5 to 8 transform the equation into one in rectangular coordinates.

5.[R] $r = \sin(\theta)$

6.[R] $r = \csc(\theta)$

7.[R] $r = 4\cos(\theta) + 5\sin(\theta)$

8.[R] $r = 3/(4\cos(\theta) + 5\sin(\theta))$

In Exercises 9 to 12 transform the equation into one in polar coordinates.

9.[R] $x = -2$

10.[R] $y = x^2$

11.[R] $xy = 1$

12.[R] $x^2 + y^2 = 4x$

In Exercises 13 to 22 graph the given equations.

13.[R] $r = 1 + \sin\theta$

14.[R] $r = 3 + 2\cos(\theta)$

15.[R] $r = e^{-\theta/\pi}$

16.[R] $r = 4^{\theta/\pi}, \theta > 0$

17.[R] $r = \cos(3\theta)$

18.[R] $r = \sin(2\theta)$

19.[R] $r = 2$

20.[R] $r = 3$

21.[R] $r = 3\sin(\theta)$

22.[R] $r = -2\cos(\theta)$

23.[M] Suppose $r = 1/\theta$ for $\theta > 0$.

(a) What happens to the y coordinate of (r, θ) as $\theta \to \infty$?

(b) What happens to the x coordinate of (r, θ) as $\theta \to \infty$?

(c) Sketch the curve.

24.[R] Suppose $r = 1/\sqrt{\theta}$ for $\theta > 0$.

(a) What happens to the y coordinate of (r, θ) as $\theta \to \infty$?

(b) What happens to the x coordinate of (r, θ) as $\theta \to \infty$?

(c) Sketch the curve.

In Exercises 25 to 30, find the intersections of the curves after drawing them.

25.[R] $r = 1+\cos(\theta)$ and $r = \cos(\theta) - 1$

26.[R] $r = \sin(2\theta)$ and $r = 1$

27.[R] $r = \sin(3\theta)$ and $r = \cos(3\theta)$

28.[R] $r = 2\sin(2\theta)$ and $r = 1$

29.[R] $r = \sin(\theta)$ and $r = \cos(2\theta)$

30.[R] $r = \cos(\theta)$ and $r = \cos(2\theta)$

A curve $r = 1 + a\cos(\theta)$ (or $r = 1 + a\sin(\theta)$) is called a **limaçon** (pronounced lee' · ma · son). Its shape depends on the choice of the constant a. For $a = 1$ we have the cardioid of Example 2. Exercises 31 to 33 concern other choices of a.

31.[R] Graph $r = 1 + 2\cos(\theta)$. (If $|a| > 1$, then the graph of $r = 1 + a\cos\theta$ crosses itself and forms two loops.)

32.[R] Graph $r = 1 + \frac{1}{2}\cos(\theta)$.

33.[C] Consider the curve $r = 1 + a\cos(\theta)$, where $0 \leq a \leq 1$.

(a) Relative to the same polar axis, graph the curves corresponding to $a = 0$, $1/4$, $1/2$, $3/4$, 1

(b) For $a = 1/4$ the graph in (a) is convex, but not for $a = 1$. Show that for $1/2 < a \leq 1$ the curve is not convex. NOTE: "Convex" is defined in Section 2.5 on page 115. HINT: Find the points on the curve farthest to the left and compare them to the point on the curve corresponding to $\theta = \pi$.

34.[M]

(a) Graph $r = 3 + \cos(\theta)$

(b) Find the point on the graph
maximum y coordinate.

35.[M] Find the y coordinate of t
the right-hand leaf of the four-leav

36.[M] Graph $r^2 = \cos(2\theta)$. Not
negative, r is not defined and that, i
there are two values of r, $\sqrt{\cos(2}$
This curve is called a **lemniscate**.

In Appendix E it is shown that
$1/(1 + e\cos(\theta))$ is a parabola if e
$0 \leq e < 1$, and a hyperbola if $e >$
"eccentricity," not Euler's number.
concern such graphs.

37.[M]

(a) Graph $r = \frac{1}{1+\cos(\theta)}$.

(b) Find an equation in rectangular coordinates for the curve in (a).

(a) (

(b)

38.[M]

39.[C] Where do the spirals $r =$
$\theta \geq 0$, intersect?

9.2 Computing Area in Polar Coordinates

In Section 6.1 we saw how to compute the area of a region if the lengths of parallel cross sections are known. Sums based on rectangles led to the formula

$$\text{Area} = \int_a^b c(x)\ dx$$

where $c(x)$ denotes the cross-sectional length. Now we consider quite a different situation, in which sectors of circles, not rectangles, provide an estimate of the area.

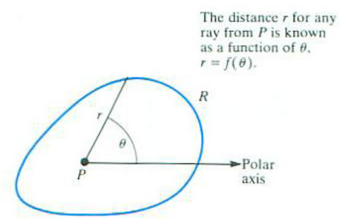


Figure 9.2.1:

Let R be a region in the plane and P a point inside it, that we take as the pole of a polar coordinate system. Assume that the distance r from P to any point on the boundary of R is known as a function $r = f(\theta)$. Also, assume that any ray from P meets the boundary of R just once, as in Figure 9.2.1.

The cross sections made by the rays from P are *not* parallel. Instead, like the spokes in a wheel, they all meet at the point P. It would be unnatural to use rectangles to estimate the area, but it is reasonable to use sectors of circles that have P as a common vertex.

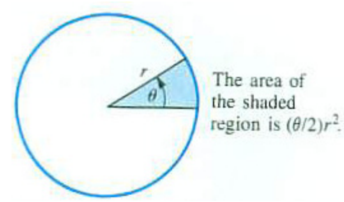


Figure 9.2.2:

Begin by recalling that in a circle of radius r a sector of central angle θ has area $(\theta/2)r^2$. (See Figure 9.2.2.) This formula plays the same role now as the formula for the area of a rectangle did in Section 6.1.

Area in Polar Coordinates

Let R be the region bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the curve $r = f(\theta)$, as shown in Figure 9.2.3. To obtain a **local estimate** for the area of R, consider the portion of R between the rays corresponding to the angles θ and $\theta + d\theta$, where $d\theta$ is a small positive number. (See Figure 9.2.4(a).) The area of the narrow wedge is shaded in Figure 9.2.4(a) is approximately that of a sector of a circle of radius $r = f(\theta)$ and angle $d\theta$, shown in Figure 9.2.4(b). The area of the sector in Figure 9.2.4(b) is

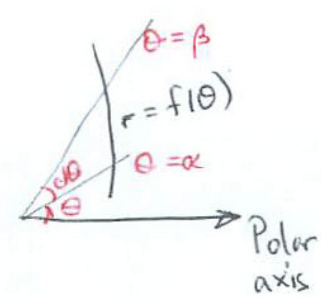


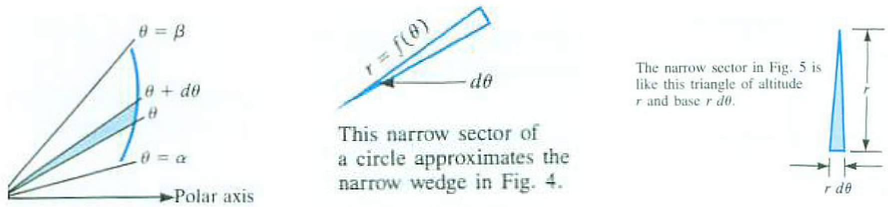
Figure 9.2.3:

$$\frac{f(\theta)^2}{2} d\theta. \tag{9.2.1}$$

Having found the local estimate of area (9.2.1), we conclude that the area of R is The area of the region bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the curve $r = f(\theta)$ is

How to find area in polar coordinates.

$$\int_\alpha^\beta \frac{f(\theta)^2}{2}\ d\theta \qquad \text{or simply} \qquad \int_\alpha^\beta \frac{r^2}{2} d\theta. \tag{9.2.2}$$



(a) (b) (c)

Figure 9.2.4:

Formula 9.2.2 is applied in Section 15.1 (and a CIE) to the motion of satellites and planets.

Area has dimensions of length squared.

Remark: It may seem surprising to find $(f(\theta))^2$, not just $f(\theta)$, in the integrand. But remember that area has the dimension "length times length." Since θ, given in radians, is dimensionless, being defined as "length of circular arc divided by length of radius", $d\theta$ is also dimensionless. Hence $f(\theta)\ d\theta$, having the dimension of length, not of area, could *not* be correct. But $\frac{1}{2}(f(\theta))^2\ d\theta$, having the dimension of area (length times length), is plausible. For rectangular coordinates, in the expressions $f(x)\ dx$, both $f(x)$ and dx have the dimension of length, one along the y-axis, the other along the x-axis; thus $f(x)\ dx$ has the dimension of area. As an aid in remembering the area of the narrow sector in Figure 9.2.4(b), note that it resembles a triangle of height r and base $r\ d\theta$, as shown in Figure 9.2.4(c). Its area is

Memory device

$$\frac{1}{2} \cdot \underbrace{r}_{\text{height}} \cdot \underbrace{rd\theta}_{\text{base}} = \frac{r^2 d\theta}{2}.$$

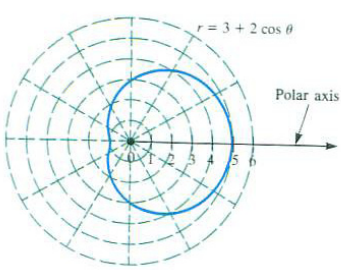


Figure 9.2.5:

EXAMPLE 1 Find the area of the region bounded by the polar curve $r = 3 + 2\cos(\theta)$, shown in Figure 9.2.5.

SOLUTION This cardiod is traced once for $0 \le \theta \le 2\pi$. By the formula just obtained, this area is

$$\int_0^{2\pi} \frac{1}{2}(3+2\cos(\theta))^2 d\theta = \frac{1}{2}\int_0^{2\pi}(9+12\cos(\theta)+4\cos^2(\theta))d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}(9+12\cos(\theta)+2(1+\cos(2\theta))\ d\theta$$
$$= \frac{1}{2}(9\theta+12\sin(\theta)+2\theta+\sin(2\theta))\Big|_0^{2\pi} = 11\pi.$$

◇

EXAMPLE 2 Find the area of the region inside one of the eight loops of the eight-leaved rose $r=\cos(4\theta)$.

SOLUTION To graph one of the loops, start with $\theta=0$. For that angle, $r=\cos(4\cdot 0)=\cos 0=1$. The point $(r,\theta)=(1,0)$ is the outer tip of a loop. As θ increases from 0 to $\pi/8$, $\cos(4\theta)$ decreases from $\cos(0)=1$ to $\cos(\pi/2)=0$. One of the eight loops is therefore bounded by the rays $\theta=\pi/8$ and $\theta=-\pi/8$. It is shown in Figure 9.2.6.

The area of this loop, which is bisected by the polar axis, is

$$\int_{-\pi/8}^{\pi/8} \frac{r^2}{2} d\theta = \int_{-\pi/8}^{\pi/8} \frac{\cos^2(4\theta)}{2} d\theta = 2\cdot\frac{1}{4}\int_0^{\pi/8}(1+\cos(8\theta))\ d\theta$$
$$= \frac{1}{2}\left(\theta+\frac{\sin(8\theta)}{4}\right)\Big|_0^{\pi/8} = \frac{1}{2}\left(\frac{\pi}{8}+\frac{\sin(\pi)}{8}\right)-0 = \frac{\pi}{16}\approx 0.19635.$$

Notice how the fact that the integrand is an even function simplifies this calculation. ◇

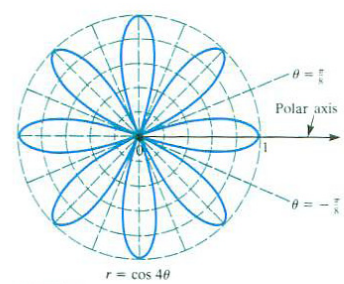


Figure 9.2.6:

The Area between Two Curves

Assume that $r=f(\theta)$ and $r=g(\theta)$ describe two curves in polar coordinates and that $f(\theta)\geq g(\theta)\geq 0$ for θ in $[\alpha,\beta]$. Let R be the region between these two curves and the rays $\theta=\alpha$ and $\theta=\beta$, as shown in Figure 9.2.7.

The area of R is obtained by subtracting the area within the inner curve, $r=g(\theta)$, from the area within the outer curve, $r=f(\theta)$.

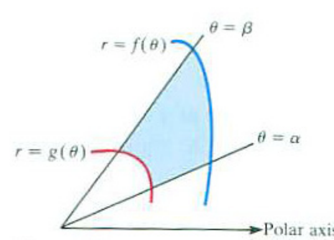


Figure 9.2.7:

EXAMPLE 3 Find the area of the top half of the region inside the cardioid $r=1+\cos(\theta)$ and outside the circle $r=\cos(\theta)$.

We must integrate over two different intervals to find the two areas.

SOLUTION The region is shown in Figure 9.2.8. The top half of the circle $r = \cos(\theta)$ is swept out as θ goes from 0 to $\pi/2$. The area of this region is

$$\frac{1}{2}\int_0^{\pi/2} \cos^2(\theta)\ d\theta = \frac{\pi}{8}.$$

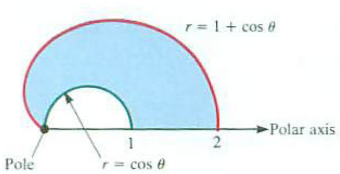


Figure 9.2.8: It's even easier to see this area as half the area of a circle of radius 1/2: $\frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$.

The top half of the cardioid is swept out by $r = 1 + \cos(\theta)$ as θ goes from 0 to π; so its area is

$$\begin{aligned}
\frac{1}{2}\int_0^{\pi}(1+\cos(\theta))^2 d\theta &= \frac{1}{2}\int_0^{\pi}\left(1 + 2\cos(\theta) + \cos^2(\theta)\right) d\theta \\
&= \frac{1}{2}\int_0^{\pi}\left(1 + 2\cos(\theta) + \frac{1+\cos(2\theta)}{2}\right) d\theta \\
&= \frac{1}{2}\int_0^{\pi}\left(\frac{3}{2} + 2\cos(\theta) + \frac{\cos(2\theta)}{2}\right) d\theta \\
&= \frac{1}{2}\left(\frac{3\theta}{2} + 2\sin(\theta) + \frac{\sin(2\theta)}{4}\right)\Big|_0^{\pi} \\
&= \frac{3\pi}{4}.
\end{aligned}$$

Thus the area in question is

$$\frac{3\pi}{4} - \frac{\pi}{8} = \frac{5\pi}{8} \approx 1.96349.$$

◇

Summary

In this section we saw how to find the area within a curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$. The heart of the method is the local approximation by a narrow sector of radius r and angle $d\theta$, which has area $r^2 d\theta/2$. (It resembles a triangle of height r and base $r d\theta$.) This approximation leads to the formula,

$$\text{Area} = \int_{\alpha}^{\beta} \frac{r^2}{2} d\theta.$$

It is more prudent to remember the triangle than the area formula because you may otherwise forget the 2 in the denominator.

EXERCISES for Section 9.2 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 6, draw the bounded region enclosed by the indicated curve and rays and then find its area.

1.[R] $r = 2\theta$, $\alpha = 0$, $\beta = \frac{\pi}{2}$

2.[R] $r = \sqrt{\theta}$, $\alpha = 0$, $\beta = \pi$

3.[R] $r = \frac{1}{1+\theta}$, $\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{2}$

4.[R] $r = \sqrt{\sin(\theta)}$, $\alpha = 0$, $\beta = \frac{\pi}{2}$

5.[R] $r = \tan(\theta)$, $\alpha = 0$, $\beta = \frac{\pi}{4}$

6.[R] $r = \sec(\theta)$, $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{4}$

In each of Exercises 7 to 16 draw the region bounded by the indicated curve and then find its area.

7.[R] $r = 2\cos(\theta)$

8.[R] $r = e^{\theta}, 0 \le \theta \le 2\pi$

9.[R] Inside the cardioid $r = 3 + 3\sin(\theta)$ and outside the circle $r = 3$.

10.[R] $r = \sqrt{\cos(2\theta)}$

11.[R] One loop of $r = \sin(3\theta)$

12.[R] One loop of $r = \cos(2\theta)$

13.[R] Inside one loop of $r = 2\cos(2\theta)$ and outside $r = 1$

14.[R] Inside $r = 1 + \cos(\theta)$ and outside $r = \sin(\theta)$

15.[R] Inside $r = \sin(\theta)$ and outside $r = \cos(\theta)$

16.[R] Inside $r = 4 + \sin(\theta)$ and outside $r = 3 + \sin(\theta)$

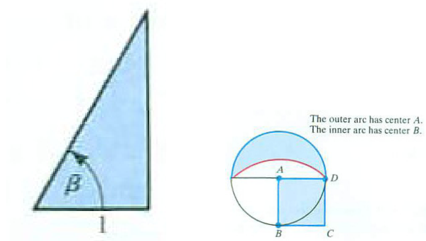
17.[M] Sketch the graph of $r = 4 + \cos(\theta)$. Is it a circle?

18.[M]

(a) Show that the area of the triangle in Figure 9.2.9(a) is $\int_0^{\beta} \frac{1}{2}\sec^2(\theta)d\theta$.

(b) From (a) and the fact that the area of a triangle is $\frac{1}{2}$(base)(height), show that $\tan(\beta) = \int_0^{\beta} \sec^2(\theta)d\theta$.

(c) With the aid of the equation in (b), obtain another proof that $(\tan(x))' = \sec^2(x)$.



(a) (b)

Figure 9.2.9:

19.[M] Show that the area of the shaded crescent between the two circular arcs is equal to the area of square $ABCD$. (See Figure 9.2.9(b).) This type of result encouraged mathematicians from the time of the Greeks to try to find a method using only straightedge and compass for constructing a square whose area equals that of a given circle. This was proved impossible at the end of the nineteenth century by showing that π is not the root of a non-zero polynomial with integer coefficients.

20.[M]

(a) Graph $r = 1/\theta$ for $0 < \theta \le \pi/2$.

(b) Is the area of the region bounded by the curve drawn in (a) and the rays $\theta = 0$ and $\theta = \pi/2$ finite or infinite?

21.[M]

(a) Sketch the curve $r = 1/(1 + \cos(\theta))$.

(b) What is the equation of the curve in (a) in rectangular coordinates?

(c) Find the area of the region bounded by the curve in (a) and the rays $\theta = 0$ and $\theta = 3\pi/4$, using polar coordinates.

(d) Solve (c) using retangular coordinates and the equation in (b).

22.[M] Use Simpson's method to estimate the area of the bounded region between $r = \sqrt[3]{1+\theta^2}$, $\theta = 0$, and $\theta = \pi/2$ that is correct to three decimal places.

23.[C] Estimate the area of the region bounded by $r = e^{\theta}$, $r = 2\cos(\theta)$ and $\theta = 0$. HINT: You may need to approximate a limit of integration.

24.[C] Figure 9.2.10 shows a point P inside a convex region $\mathcal{R}$.

(a) Assume that P cuts each chord through P into two intervals of equal length. Must each chord through P cut $\mathcal{R}$ into two regions of equal areas?

(b) Assume that each chord through P cuts $\mathcal{R}$ into two regions of equal areas. Must P cut each chord through P into two intervals of equal lengths?

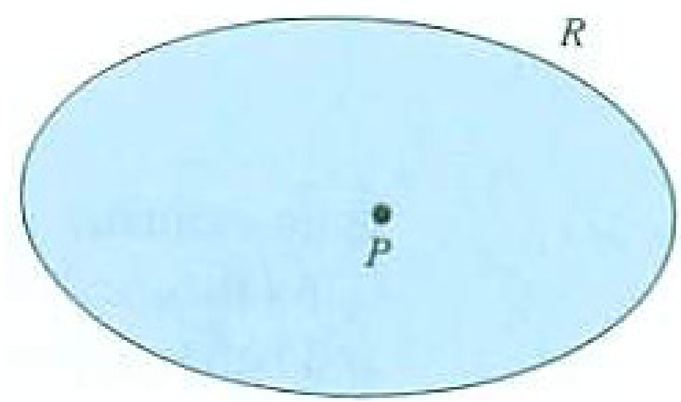


Figure 9.2.10:

25.[C] Let $\mathcal{R}$ be a convex region in the plane and P be a point on the boundary of $\mathcal{R}$. Assume that every chord of $\mathcal{R}$ that has an end at P has length at least 1.

(a) Draw several examples of such an $\mathcal{R}$.

(b) Make a general conjecture a

(c) Prove it.

26.[C] Repeat Exercise 25, exce through P has length not more th

27.[C]

(a) Show that each line through the region bounded by the in a segment of length 6.

(b) Each line through the cente 3 also intersects the disk in 6. Does it follow that the di Example 1 have the same a

28.[C] Consider a convex region point P inside it. If you know the that passes through P. Can you area of $\mathcal{R}$

(a) if P is on the border of $\mathcal{R}$?

(b) if P is in the interior of $\mathcal{R}$?

Exercises 29 to 31, contributed by lated.

29.[C] The graph of $r = \cos(n\theta)$ n is even. Find the total area with

30.[C] The graph of $r = \cos(n\theta)$ is odd. Find the total area within

31.[C] Find the total area of a the curve $r = \sin(n\theta)$, where n i HINT: Take the cases n even or od

9.3 Parametric Equations

Up to this point we have considered curves described in three forms: "y is a function of x", "x and y are related implicitly", and "r is a function of θ". But a curve is often described by giving both x and y as functions of a third variable. We introduce this situation as it arises in the study of motion. It was the basis for the CIE on the Uniform Sprinkler in Chapter 5.

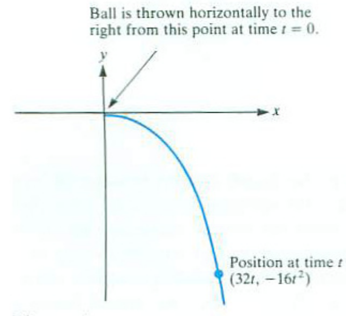


Figure 9.3.1:

Two Examples

EXAMPLE 1 If a ball is thrown horizontally out of a window with a speed of 32 feet per second, it falls in a curved path. Air resistance disregarded, its position t seconds later is given by $x = 32t$, $y = -16t^2$ relative to the coordinate system in Figure 9.3.1. Here the curve is completely described, not by expressing y as a function of x, but by expressing each of x and y as functions of a third variable t. The third variable is called a **parameter**. The equations $x = 32t$, $y = -16t^2$ are called **parametric equations** for the curve.

para meaning "together," *meter* meaning "measure".

In this example it is easy to eliminate t and so find a direct relation between x and y:

$$t = \frac{x}{32}.$$

Hence

$$y = -16\left(\frac{x}{32}\right)^2 = -\frac{16}{(32)^2}x^2 = -\frac{1}{64}x^2.$$

The path is part of the parabola $y = -\frac{1}{64}x^2$. ◇

In Example 2 elimination of the parameter would lead to a complicated equation involving x and y. One advantage of parametric equations is that they can provide a simple description of a curve, although it may be impossible to find an equation in x and y that describes the curve.

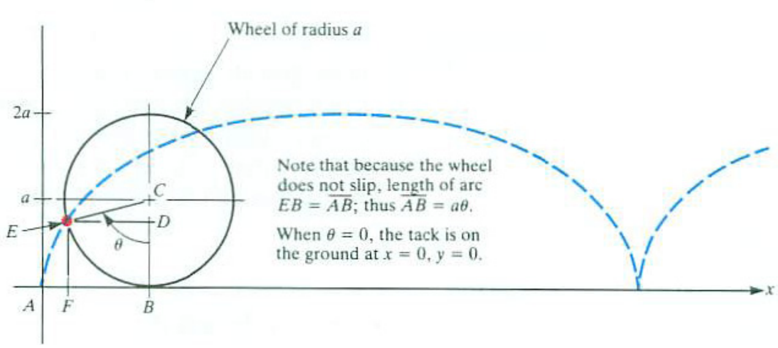


Figure 9.3.2:

EXAMPLE 2 As a bicycle wheel of radius a rolls along, a tack stuck in its circumference traces out a curve called a **cycloid**, which consists of a sequence of arches, one arch for each revolution of the wheel. (See Figure 9.3.2.) Find the position of the tack as a function of the angle θ through which the wheel turns.

SOLUTION Assume that the tack is initially at the bottom of the wheel. The x coordinate of the tack, corresponding to θ, is

$$|\overline{AF}| = |\overline{AB}| - |\overline{ED}| = a\theta - a\sin(\theta),$$

and the y coordinate is

$$|\overline{EF}| = |\overline{BC}| - |\overline{CD}| = a - a\cos(\theta).$$

Then the position of the tack, as a function of the parameter θ, is

$$x = a\theta - a\sin(\theta), \quad y = a - a\cos(\theta).$$

See Exercise 36.

In this case, eliminating θ leads to a complicated relation between x and y. ◇

Any curve can be described parametrically. For instance, consider the curve $y = e^x + x$. It is perfectly legal to introduce a parameter t equal to x and write

$$x = t, \quad y = e^t + t.$$

This device may seem a bit artificial, but it will be useful in the next section in order to apply results for curves expressed by means of parametric equations to curves given in the form $y = f(x)$.

How to Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

How can we find the slope of a curve that is described parametrically by the equations

$$x = g(t), \quad y = h(t)?$$

An often difficult, perhaps impossible, approach is to solve the equation $x = g(t)$ for t as a function of x and substitute the result into the equation $y = h(t)$, thus expressing y explicitly in terms of x; then differentiate the result to find dy/dx. Fortunately, there is a very easy way, which we will now describe. Assume that y is a differentiable function of x. Then, by the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

from which it follows that

Slope of a parameterized curve

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{9.3.1}$$

It is assumed that in formula (9.3.1) dx/dt is not 0. To obtain d^2y/dx^2 just replace y in (9.3.1) by dy/dx, obtaining

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

EXAMPLE 3 At what angle does the arch of the cycloid shown in Example 2 meet the x-axis at the origin?
SOLUTION The parametric equations of the cycloid are

$$x = a\theta - a\sin(\theta) \qquad \text{and} \qquad y = a - a\cos(\theta).$$

Here θ is the parameter. Then

$$\frac{dx}{d\theta} = a - a\cos(\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = a\sin(\theta).$$

Consequently,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin(\theta)}{a - a\cos(\theta)} = \frac{\sin(\theta)}{1-\cos(\theta)}.$$

When $\theta = 0$, $(x, y) = (0, 0)$ and $\frac{dy}{dx}$ is not defined because $\frac{dx}{d\theta} = 0$. But, when θ is near 0, (x, y) is near the origin and the slope of the cycloid at $(0, 0)$ can be found by looking at the limit of the slope, which is $\sin\theta/(1-\cos(\theta))$, as $\theta \to 0^+$. L'Hôpital's Rule applies, and we have

$$\lim_{\theta\to 0^+} \frac{\sin(\theta)}{1-\cos(\theta)} = \lim_{\theta\to 0^+} \frac{\cos(\theta)}{\sin(\theta)} = \infty.$$

Thus the cycloid comes in vertically at the origin, as shown in Figure 9.3.2. ⋄

EXAMPLE 4 Find d^2y/dx^2 for the cycloid of Example 2.
SOLUTION In Example 3 we found

$$\frac{dy}{dx} = \frac{\sin(\theta)}{1-\cos(\theta)}.$$

As shown in Example 3, $dx/d\theta = a - a\cos(\theta)$. To find $\frac{d^2y}{dx^2}$ we first compute

$$\frac{d}{d\theta}\left(\frac{dy}{dx}\right) = \frac{(1-\cos(\theta))\cos(\theta) - \sin(\theta)(\sin(\theta))}{(1-\cos(\theta))^2} = \frac{\cos(\theta)-1}{(1-\cos(\theta))^2} = \frac{-1}{1-\cos(\theta)}.$$

Thus

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{-1}{1-\cos(\theta)}}{a - a\cos(\theta)} = \frac{-1}{a(1-\cos(\theta))^2}.$$

Since the denominator is positive (or 0), the quotient, when defined, is negative. This agrees with Figure 9.3.2, which shows each arch of the cycloid as concave down. ◇

Summary

This section described parametric equations, where x and y are given as functions of a third variable, often time (t) or angle (θ). We also showed how to compute dy/dx and d^2y/dx^2:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

and replacing y by $\frac{dy}{dx}$,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

EXERCISES for Section 9.3 *Key:* R–routine, M–moderate, C–challenging

1.[R] Consider the parametric equations $x = 2t + 1$, $y = t - 1$.

(a) Fill in this table:

t	-2	-1	0	1	2
x					
y					

(b) Plot the five points (x, y) obtained in (a).

(c) Graph the curve given by the parametric equations $x = 2t + 1$, $y = t - 1$.

(d) Eliminate t to find an equation for the curve involving only x and y.

2.[R] Consider the parametric equations $x = t + 1$, $y = t^2$.

(a) Fill in this table:

t	-2	-1	0	1	2
x					
y					

(b) Plot the five points (x, y) obtained in (a).

(c) Graph the curve.

(d) Find an equation in x and y that describes the curve.

3.[R] Consider the parametric equations $x = t^2$, $y = t^2 + t$.

(a) Fill in this table:

t	-3	-2	-1	0	1	2	3
x							
y							

(b) Plot the seven points (x, y) obtained in (a).

(c) Graph the curve given by $x = t^2$, $y = t^2 + t$.

(d) Eliminate t and find an equation for the graph in terms of x and y.

4.[R] Consider the parametric equations $x = 2\cos(t)$, $y = 3\sin(t)$.

(a) Fill in this table, expressing the entries decimally:

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x									
y									

(b) Plot the eight distinct points in *(a)*.

(c) Graph the curve given by $x = 2\cos(t)$, $y = 3\sin(t)$.

(d) Using the identity $\cos^2(t) + \sin^2(t) = 1$, eliminate t.

In Exercises 5 to 8 express the curves parametrically with parameter t.

5.[R] $y = \sqrt{1 + x^3}$

6.[R] $y = \tan^{-1}(3x)$

7.[R] $r = \cos^2(\theta)$

8.[R] $r = 3 + \cos(\theta)$

In Exercises 9 to 14 find dy/dx and d^2y/dx^2 for the given curves.

9.[R] $x = t^3 + t$, $y = t^7 + t + 1$

10.[R] $x = \sin(3t)$, $y = \cos(4t)$

11.[R] $x = 1 + \ln(t)$, $y = t\ln(t)$

12.[R] $x = e^{t^2}$, $y = \tan(t)$

13.[R] $r = \cos(3\theta)$

14.[R] $r = 2 + 3\sin(\theta)$

In Exercises 15 to 16 find the equation of the tangent line to the given curve at the given point.

15.[R] $x = t^3 + t^2$, $y = \sec 3t$; $(1, 1)$
$y = t^5 + t$; $(2, 2)$
16.[R] $x = \frac{t^2+1}{t^3+t^2+1}$,

In Exercises 17 and 18 find d^2y/dx^2.

17.[R] $x = t^3 + t + 1$, $y = e^{3t} + \cos(t^2)$
$y = t^2 + t + 2$
18.[R] $x = e^{3t} + \sin(2t)$,

19.[R] For which values of t is the curve in Exercise 17 concave up? concave down?

20.[R] Let $x = t^3 + 1$ and $y = t^2 + t + 1$. For which values of t is the curve concave up? concave down?

21.[R] Find the slope of the three-leaved rose, $r = \sin(3\theta)$, at the point $(r, \theta) = (\sqrt{2}/2, \pi/12)$.

22.[R]

(a) Find the slope of the cardioid $r = 1 + \cos(\theta)$ at the point (r, θ).

(b) What happens to the slope in (a) as θ approaches π from the left?

(c) What does (b) tell us about the graph of the cardioid? (Show it on the graph.)

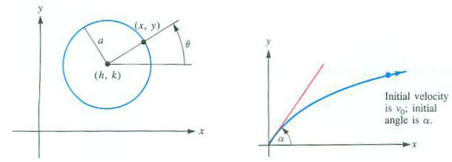
23.[R] Obtain parametric equations for the circle of radius a and center (h, k), using as parameter the angle θ shown in Figure 9.3.3(a).

24.[R] At time $t \geq 0$ a ball is at the point $(24t, -16t^2 + 5t + 3)$.

(a) Where is it at time $t = 0$?

(b) What is its horizontal speed at that time?

(c) What is its vertical speed at that time?



(a) (b)

Figure 9.3.3:

Exercises 25 to 27 analyze the trajectory of a ball thrown from the origin at an angle α and initial velocity v_0, as sketched in Figure 9.3.3(b). These results are used in the CIE on the Uniform Sprinkler in Chapter 5 (see page 412).

25.[R] It can be shown that if time is in seconds and distance in feet, then t seconds later the ball is at the point

$$x = (v_0 \cos(\alpha))t, \quad y = (v_0 \sin(\alpha))t - 16t^2.$$

(a) Express y as a function of x. Hint: Eliminate t.

(b) In view of (a), what type of curve does the ball follow?

(c) Find the coordinates of its highest point.

26.[R] Eventually the ball in Exercise 25 falls back to the ground.

(a) Show that, for a given v_0, the horizontal distance it travels is proportional to $\sin(2\theta)$.

(b) Use (a) to determine the angle that maximizes the horizontal distance traveled.

(c) Show that the horizontal distance traveled in (a) is the same when the ball is thrown at an angle of θ or at an angle of $\pi/2 - \theta$.

27.[R] Is it possible to extend the horizontal distance traveled by throwing the ball in Exercise 25 from the top of a hill? (Assume the hill has height d.) HINT: Work with the horizontal distance traveled, x, not the distance along the sloped ground.

28.[R] The spiral $r = e^{2\theta}$ meets the ray $\theta = \alpha$ at an infinite number of points.

(a) Graph the spiral.

(b) Find the slope of the spiral at each intersection with the ray.

(c) Show that at all of these points this spiral has the same slope.

(d) Show that the analog of (c) is not true for the spiral $r = \theta$.

29.[M] The spiral $r = \theta$, $\theta > 0$ meets the ray $\theta = \alpha$ at an infinite number of points (α, α), $(\alpha + 2\pi, \alpha)$, $(\alpha + 4\pi, \alpha)$, ... What happens to the angle between the spiral and the ray at the point $(\alpha + 2\pi n, \alpha)$ as $n \to \infty$?

30.[M] Let a and b be positive numbers. Consider the curve given parametrically by the equations

$$x = a\cos(t) \qquad y = b\sin(t).$$

(a) Show that the curve is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(b) Find the area of the region bounded by the ellipse in (a) by making a substitution that expresses $4\int_0^a y\,dx$ in terms of an integral in which the variable is t and the range of integration is $[0, \pi/2]$.

31.[M] Consider the curve given parametrically by

$$x = t^2 + e^t \qquad y = t + e^t$$

for t in $[0, 1]$.

(a) Plot the points corresponding to $t = 0$, $1/2$, and 1.

(b) Find the slope of the curve at the point $(1, 1)$.

(c) Find the area of the region under the curve and above the interval $[1, e+1]$. [See Exercise 30(b).]

32.[M] What is the slope of the cycloid in Figure 9.3.2 at the first point on it to the right of the y-axis at the height a?

33.[M] The region under the arch of the cycloid

$$x = a\theta - a\sin(\theta), \quad y = a - a\cos(\theta) \qquad (0 \le \theta \le 2\pi)$$

and above the x-axis is revolved around the x-axis. Find the volume of the solid of revolution produced.

34.[M] Find the volume of the solid of revolution obtained by revolving the region in Exercise 33 about the y-axis.

35.[M] Let a be a positive constant. Consider the curve given parametrically by the equations $x = a\cos^3(t)$, $y = a\sin^3(t)$.

(a) Sketch the curve.

(b) Express the slope of the curve in terms of the parameter t.

36.[M] Solve the parametric equations for the cycloid, $x = a\theta - a\sin(\theta)$, $y = a - a\cos(\theta)$, for y as a function of x. NOTE: See Example 1.

37.[C] Consider a tangent line to the curve in Exercise 35 at a point P in the first quadrant. Show that the length of the segment of that line intercepted by the coordinate axes is a.

38.[C] L'Hôpital's rule in Section 5.5 asserts that if $\lim_{t\to 0} f(t) = 0$, $\lim_{t\to 0} g(t) = 0$, and $\lim_{t\to 0}(f'(t)/g'(t))$ exists, then $\lim_{t\to 0}(f(t)/g(t)) = \lim_{t\to 0}(f'(t)/g'(t))$. Interpret that rule in terms of the parameterized curve $x = g(t)$, $y = f(t)$. HINT: Make a sketch of the curve near $(0, 0)$ and show on it the geometric meaning of the quotients $f(t)/g(t)$ and $f'(t)/g'(t)$.

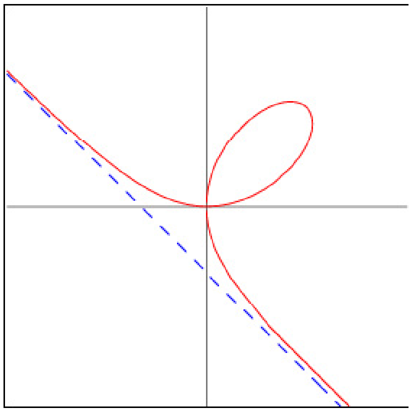


Figure 9.3.4:

39.[C] The **Folium of Descartes** is the graph of

$$x^3 + y^3 = 3xy.$$

The graph is shown in Figure 9.3.4. It consists of a loop and two infinite pieces both line $x + y + 1 = 0$. Parameterize th t of the line joining the origin with point (x, y) on the curve, $y = xt$.

(a) Show that

$$x = \frac{3t}{1+t^3} \qquad \text{and}$$

(b) Find the highest point on th

(c) Find the point on the loop f

(d) The loop is parameterized by values of t parameterize the quadrant?

(e) Which values of t parameter second quadrant?

(f) Show that the Folium of Des with respect to the line $y =$

NOTE: Visit `http://en.wik Folium_of_Descartes` or do a "Folium Descartes" to see its lon back to 1638.

9.4 Arc Length and Speed on a Curve

In Section 4.2 we studied the motion of an object moving on a line. If at time t its position is $x(t)$, then its velocity is the derivative $\frac{dx}{dt}$ and its speed is $\left|\frac{dx}{dt}\right|$. Now we will examine the velocity and speed of an object moving along a curved path.

Arc Length and Speed in Rectangular Coordinates

Consider an object moving on a path given parametrically by

$$\begin{cases} x &= g(t) \\ y &= h(t) \end{cases},$$

where g and h have continuous derivatives. Think of t as time, though the parameter could be anything, such as angle or even x itself.

First, let us find a formula for its speed.

Let $s(t)$ be the arc length covered from the initial time to an arbitrary time t. In a short interval of time, Δt, it travels a distance Δs along the path. We want to find

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

We take an intuitive approach, and leave a more formal argument for Exercise 30.

During the time interval $[t, t + \Delta t]$ the object goes from P to Q on the path, covering a distance Δs, as shown in Figure 9.4.1. During this time its x-coordinate changes by Δx and its y-coordinate by Δy. The chord $\overline{PQ}$ has length $\sqrt{(\Delta x)^2 + (\Delta y)^2}$.

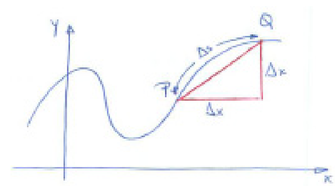


Figure 9.4.1:

We assume now that the curve is well behaved in the sense that $\lim_{\Delta t \to 0} \frac{\Delta s}{|\overline{PQ}|} = 1$. In this case,

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} &= \lim_{\Delta t \to 0} \frac{|\overline{PQ}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} \\ &= \lim_{\Delta t \to 0} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \end{aligned}$$

We have just obtained the key result in this section:

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

or, stated in terms of differentials,

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

The rates at which x and y change determine how fast the arc length s changes, as recorded in Figure 9.4.2.

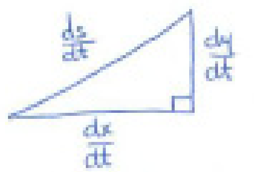


Figure 9.4.2:

Now that we have a formula for ds/dt, we simply integrate it to get the distance along the path covered during a time interval $[a, b]$:

$$\text{arc length} = \int_a^b ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt. \qquad (9.4.1)$$

If the curve is given in the form $y = f(x)$, one is free to use x as the parameter. Thus, a parametric representation of the curve is

$$x = x, \qquad y = f(x).$$

Then (9.4.1) becomes

$$\text{arc length} = \int_a^b \sqrt{1 + (f'(x))^2}\ dx.$$

WARNING *(Sign of $\frac{ds}{dt}$)* The arclength function is, by definition, an non-decreasing function. This means ds/dt is never negative. In fact, in most applications ds/dt will be strictly positive.

Three examples will show how these formulas are applied. The first goes back to the year 1657, when the 20-year old Englishman, William Neil, found the length of an arc on the graph of $y = x^{3/2}$. His method was much more complicated. Earlier in that century, Thomas Harriot had found the length of an arc of the spiral $r = e^{\theta}$, but his work was not widely published.

EXAMPLE 1 Find the arc length of the curve $y = x^{3/2}$ for x in $[0, 1]$. (See Figure 9.4.3.)

SOLUTION By formula (9.4.1),

$$\text{arc length} = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

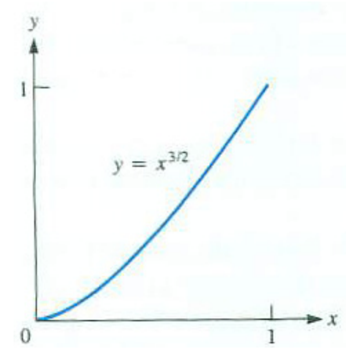


Figure 9.4.3:

Since $y = x^{3/2}$, we differentiate to find $dy/dx = \frac{3}{2}x^{1/2}$. Thus

$$\begin{aligned}\text{arc length} &= \int_0^1 \sqrt{1+\left(\tfrac{3}{2}x^{1/2}\right)^2}\,dx = \int_0^1 \sqrt{1+\tfrac{9}{4}x}\,dx \\ &= \int_1^{13/4} \sqrt{u}\cdot\tfrac{4}{9}\,du \qquad (u = 1+\tfrac{9}{4}x,\ du = \tfrac{9}{4}dx) \\ &= \tfrac{4}{9}\cdot\tfrac{2}{3}u^{3/2}\Big|_1^{13/4} = \tfrac{8}{27}\left(\left(\tfrac{13}{4}\right)^{3/2} - 1^{3/2}\right) \\ &= \tfrac{8}{27}\left(\tfrac{13^{3/2}}{8} - 1\right) = \tfrac{13^{3/2}-8}{27} \approx 1.43971.\end{aligned}$$

◇

Incidentally, the arc length of the curve $y = x^a$ where a is a non-zero rational number, usually *cannot* be computed with the aid of the Fundamental Theorem of Calculus. The only cases in which it can be computed by the FTC are a=1 (the graph of $y = x$) and $a = 1 + \frac{1}{n}$ where n is an integer. Exercise 32 treats this question.

EXAMPLE 2 In Section 9.3 the parametric equations for the motion of a ball thrown horizontally with a speed of 32 feet per second (≈ 21.8 *mph*) were found to be $x = 32t$, $y = -16t^2$. (See Example 1 and Figure 9.3.1.) How fast is the ball moving at time t? Find the distance s which the ball travels during the first b seconds.

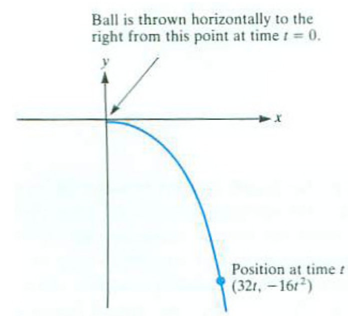


Figure 9.4.4:

SOLUTION From $x = 32t$ and $y = -16t^2$ we compute $\frac{dx}{dt} = 32$ and $\frac{dy}{dt} = -32t$. Its speed at time t is

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(32)^2 + (-32t)^2} = 32\sqrt{1+t^2} \text{ feet per second.}$$

The distance traveled is the arc length from $t = 0$ to $t = b$. By formula (9.4.1),

$$\text{arc length} = \int_0^b \sqrt{(32)^2 + (-32t)^2}\,dt = 32\int_0^b \sqrt{1+t^2}\,dt.$$

This integral can be evaluated with an integration table or with the trigonometric substitution $x = \tan(\theta)$. An antiderivative is

See Formula 31 in the integral table.

$$\frac{1}{2}\left(t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right)$$

and the distance traveled is

$$16b\sqrt{1+b^2}+16\ln\left(b+\sqrt{1+b^2}\right).$$

◇

EXAMPLE 3 Find the length of one arch of the cycloid found in Example 2 of Section 9.3.

SOLUTION Here the parameter is θ, $x = a\theta - a\sin(\theta)$, and $y = a - a\cos(\theta)$. To complete one arch of the cycloid, θ varies from 0 to 2π.

We compute

$$\frac{dx}{d\theta} = a - a\cos(\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = a\sin(\theta).$$

The square of the speed is

$$\begin{aligned}(a-a\cos(\theta))^2+(a\sin(\theta))^2 &= a^2\left((1-\cos(\theta))^2+(\sin(\theta))^2\right)\\ &= a^2\left(1-2\cos(\theta)+(\cos(\theta))^2+(\sin(\theta))^2\right)\\ &= a^2\left(2-2\cos(\theta)\right)\\ &= 2a^2(1-\cos(\theta)).\end{aligned}$$

Using boxed formula (9.4.1) and the trigonometric identity $1-\cos(\theta) = 2\sin^2(\theta/2)$, we have

$$\begin{aligned}\text{the length of one arch} &= \int_0^{2\pi}\sqrt{2a^2(1-\cos(\theta))}\,d\theta = a\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos(\theta)}\,d\theta\\ &= a\sqrt{2}\int_0^{2\pi}\sqrt{2}\sin\left(\frac{\theta}{2}\right)\,d\theta = 2a\int_0^{2\pi}\sin\left(\frac{\theta}{2}\right)\,d\theta\\ &= 2a\left(-2\cos\left(\frac{\theta}{2}\right)\Big|_0^{2\pi}\right) = 2a\left(-2(-1)-(-2)(1)\right) = 8a.\end{aligned}$$

This means that while θ varies from 0 to 2π, a bicycle travels a distance of $2\pi a \approx 6.28318a$ and a tack in the tread of the tire travels a distance $8a$. ◇

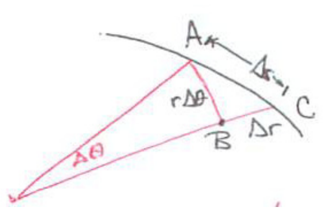


Figure 9.4.5:

Arc Length and Speed in Polar Coordinates

So far in this section curves have been described in rectangular coordinates. Next consider a curve given in polar coordinates by the equation $r = f(\theta)$.

We will estimate the length of arc Δs corresponding to small changes $\Delta\theta$ and Δr in polar coordinates, as shown in Figure 9.4.5. The region bounded by the circular arc AB, the straight segment BC, and AC, the part of the curve, resembles a right triangle whose two legs have lengths $r\Delta\theta$ and Δr. We assume Δs is well approximated by its hypotenuse, $\sqrt{(r\Delta\theta)^2 + (\Delta r)^2}$. Thus we expect

$$\begin{aligned}\frac{ds}{d\theta} = \lim_{\Delta\theta\to 0}\frac{\Delta s}{\Delta\theta} &= \lim_{\Delta\theta\to 0}\frac{\sqrt{(r\Delta\theta)^2 + (\Delta r)^2}}{(\Delta\theta)} \\ &= \lim_{\Delta\theta\to 0}\sqrt{r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2} \\ &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\end{aligned}$$

In short,

arc length for $r = f(\theta)$.

For a curve given in polar coordinates:

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \quad \text{or} \quad ds = \sqrt{(r\ d\theta)^2 + (dr)^2} = \sqrt{r^2 + (r')^2}\ d\theta.$$

This formula can also be obtained from the formula for the case of rectangular coordinates by using $x = r\cos(\theta)$ and $y = r\sin(\theta)$. (See Exercise 19.) However, we prefer the geometric approach because it is (i) more direct, (ii) more intuitive, and (iii) easier to remember.

See Exercise 19.

Arc Length of a Polar Curve $r = f(\theta)$

The length of the curve $r = f(\theta)$ for θ in $[\alpha, \beta]$ is $s = \int_\alpha^\beta\ ds$ where

$$ds = \sqrt{r^2 + (r'))^2}\ d\theta = \sqrt{(f(\theta))^2 + (f'(\theta))^2}\ d\theta.$$

EXAMPLE 4 Find the length of the spiral $r = e^{-3\theta}$ for θ in $[0, 2\pi]$.
SOLUTION First compute

$$r' = \frac{dr}{d\theta} = -3e^{-3\theta},$$

and then use the formula

$$\begin{aligned}
\text{Arc Length} &= \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\theta = \int_{0}^{2\pi} \sqrt{\left(e^{-3\theta}\right)^2 + \left(-3e^{-3\theta}\right)^2}\, d\theta \\
&= \int_{0}^{2\pi} \sqrt{e^{-6\theta} + 9e^{-6\theta}}\, d\theta = \sqrt{10} \int_{0}^{2\pi} \sqrt{e^{-6\theta}}\, d\theta \\
&= \sqrt{10} \int_{0}^{2\pi} e^{-3\theta}\, d\theta = \sqrt{10}\, \frac{e^{-3\theta}}{-3}\bigg|_{0}^{2\pi} \\
&= \sqrt{10} \left(\frac{e^{-3\cdot 2\pi}}{-3} - \frac{e^{-3\cdot 0}}{-3} \right) = \sqrt{10} \left(\frac{e^{-6\pi}}{-3} + \frac{1}{3} \right) \\
&= \frac{\sqrt{10}}{3} \left(1 - e^{-6\pi}\right) \approx 1.054093.
\end{aligned}$$

◇

Summary

This section concerns speed along a parametric path and the length of the path. If the path is described in rectangular coordinates, then Figure 9.4.6(a) conveys the key ideas. If in polar coordinates, Figure 9.4.6(b) is the key. It is much easier to recall these diagrams than the various formulas for speed and arc length. Everything depends on our old friend: the Pythagorean Theorem.

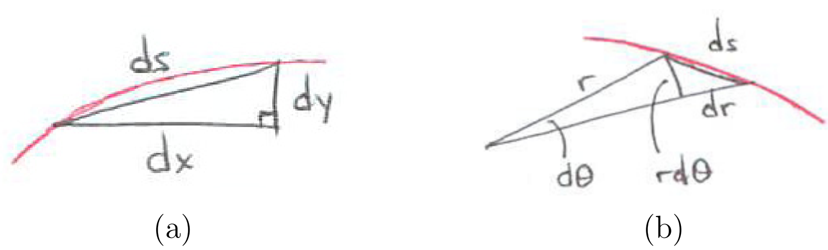


Figure 9.4.6: (a) $ds = \sqrt{(dx)^2 + (dy)^2}$ (b) $ds = \sqrt{(rd\theta)^2 + (dr)^2}$

EXERCISES for Section 9.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 8 find the arc lengths of the given curves over the given intervals.

1.[R] $y = x^{3/2}$, x in $[1, 2]$

2.[R] $y = x^{2/3}$, x in $[0, 1]$

3.[R] $y = (e^x + e^{-x})/2$, x in $[0, b]$

4.[R] $y = x^2/2 - (\ln(x))/4$, x in $[2, 3]$

5.[R] $x = \cos^3(t)$, $y = \sin^3(t)$, t in $[0, \pi/2]$

6.[R] $r = e^\theta$, θ in $[0, 2\pi]$

7.[R] $r = 1 + \cos(\theta)$, θ in $[0, \pi]$

8.[R] $r = cos^2(\theta/2)$, θ in $[0, \pi]$

In each of Exercises 9 to 12 find the speed of the particle at time t, given the parametric description of its path.

9.[R] $x = 50t$, $y = -16t^2$

10.[R] $x = \sec(3t)$, $y = \sin^{-1}(4t)$

11.[R] $x = t + \cos(t)$, $y = 2t - \sin(t)$

12.[R] $\csc(\theta/2)$, $y = \tan^{-1}(\sqrt{t})$

13.[R]

(a) Graph $x = t^2$, $y = t$ for $0 \leq t \leq 3$.

(b) Estimate its arc length from $(0, 0)$ to $(9, 3)$ by an inscribed polygon whose vertices have x-coordinates 0, 1, 4, and 9.

(c) Set up a definite integral for the arc length of the curve in question.

(d) Estimate the definite integral in (c) by using a partition of $[0, 3]]$ into 3 sections, each of length 1, and the trapezoid method.

(e) Estimate the definite integral in (c) by Simpson's method with six sections.

(f) If your calculator has a program to evaluate definite integrals, use it to evaluate the definite integral in (c) to four decimal places.

14.[R]

(a) Graph $y = 1/x^2$ for x in $[1, 2]$.

(b) Estimate the length of the arc in (a) by using an inscribed polygon whose vertices at $(1, 1)$, $(\frac{5}{4}, (\frac{4}{5})^2)$, $(\frac{3}{2}, (\frac{2}{3})^2)$, and $(2, \frac{1}{4})$.

(c) Set up a definite integral for the arc length of the curve in question.

(d) Estimate the definite integral in (c) by the trapezoid method, using four equal length sections.

(e) Estimate the definite integral in (c) by Simpson's method with four sections.

(f) If your calculator has a program to evaluate definite integrals, use it to evaluate the definite integral in (c) to four decimal places.

15.[R] How long is the spiral $r = e^{-3\theta}$, $\theta \geq 0$?

16.[R] How long is the spiral $r = 1/\theta$, $\theta \geq 2\pi$?

17.[R] Assume that a curve is described in rectangular coordinates in the form $x = f(y)$. Show that

$$\text{Arc Length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

where y ranges in the interval $[c, d]$, using a little triangle whose sides have length dx, dy, and ds.

18.[R] Consider the arc length of the curve $y = x^{2/3}$ for x in the interval $[1, 8]$.

(a) Set up a definite integral for this arc length using x as the parameter.

(b) Set up a definite integral for this arc length using y as the parameter.

(c) Evaluate the easier of the two integrals found in parts (a) and (b).

NOTE: See Exercise 17.

19.[M] We obtained the formula $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$ geometrically.

(a) Obtain the same result by calculus, starting with $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$, and using the relations $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

(b) Which derivation do you prefer? Why?

20.[M] Let $P = (x, y)$ depend on θ as shown in Figure 9.4.7.

(a) Sketch the curve that P sweeps out.

(b) Show that $P = (2\cos(\theta), \sin(\theta))$.

(c) Set up a definite integral for the length of the curve described in P. (Do not evaluate it.)

(d) Eliminate θ and show that P is on the ellipse

$$\frac{x^2}{4} + \frac{y^2}{1} = 1.$$

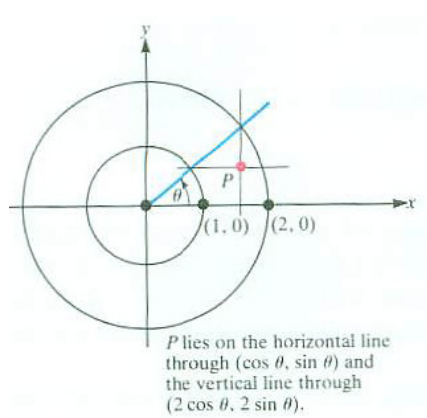


Figure 9.4.7:

21.[M]

(a) At time t a particle has polar coordinates $r = g(t)$, $\theta = h(t)$. How fast is it moving?

(b) Use the formula in (a) to find the speed of a particle which at time t is at the point $(r, \theta) = (e^t, 5t)$.

22.[M]

(a) How far does a bug travel from time $t = 1$ to time $t = 2$ if at time t it is at the point $(x, y) = (\cos\pi t, \sin\pi t)$?

(b) How fast is it moving at time t?

(c) Graph its path relative to an xy coordinate system. Where is it at time $t = 1$? At $t = 2$?

(d) Eliminate t to find a relation between x and y.

23.[M] Find the arc length of the Archimedean spiral $r = a\theta$ for θ in $[0, 2\pi]$ if a is a positive constant.

24.[M] Consider the cardioid $r = 1 + \cos\theta$ for θ in $[0, \pi]$. We may consider r as a function of θ or as a function of s, arc length along the curve, measured, say, from $(2, 0)$.

(a) Find the average of r with respect to θ in $[0, \pi]$.

(b) Find the average of r with respect to s. HINT: Express all quantities appearing in this average in terms of θ.

(See also Exercises 13 and 14 in the Chapter 9 Summary.)

25.[M] Let $r = f(\theta)$ describe a curve in polar coordinates. Assume that $df/d\theta$ is continuous. Let θ be a function of time t. Let $s(t)$ be the length of the curve corresponding to the time interval $[a, t]$.

(a) What definite integral is equal to $s(t)$?

(b) What is the speed ds/dt?

26.[M] The function $r = f(\theta)$ describes, for θ in $[0, 2\pi]$, a curve in polar coordinates. Assume r' is continuous and $f(\theta) > 0$. Prove that the average of r as a function of arc length is at least as large as the quotient $2A/s$, where A is the area swept out by the radius and s is the arc length of the curve. For which curve is the average equal to $2A/s$?

27.[M] The equations $x = \cos t$, $y = 2\sin t$, t in $[0, \pi/2]$ describe a quarter of an ellipse. Draw this arc. Describe at least two different ways of estimating the length of this arc. Compare the advantages and challenges each method presents. Use the method of your choice to estimate the length of this arc.

28.[M] When a curve is given in rectangular coordinates, its slope is $\frac{dy}{dx}$. To find the slope of the tangent line to the curve given in polar coordinates involves a bit more work.
Assume that $r = f(\theta)$. To begin use the relation

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta},$$

which is the Chain Rule in disguise ($\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$).

(a) Using the equations $y = r\sin(\theta)$ and $x = r\cos(\theta)$, find $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$.

(b) Show that the slope is

$$\frac{r\cos(\theta) + \frac{dr}{d\theta}\sin(\theta)}{-r\sin(\theta) + \frac{dr}{d\theta}\cos(\theta)}. \qquad (9.4.2)$$

29.[M] Use (9.4.2) to find the slope of the cardioid $r = 1 + \sin(\theta)$ at $\theta = \frac{\pi}{3}$.

30.[M] Show that if $\lim_{\Delta t \to 0} \frac{\Delta s}{|\overline{PQ}|} = 1$, then $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{|\overline{PQ}|}{\Delta t}$.

31.[C] Let $y = f(x)$ for x in $[0, 1]$ describe a curve that starts at $(0, 0)$, ends at $(1, 1)$, and lies in the square with vertices $(0, 0)$,$(1, 0)$,$(1, 1)$, and $(0, 1)$. Assume f has a continuous derivative.

(a) What can be said about the arc length of the curve? How small and how large can it be?

(b) Answer (a) if it is assumed also that $f'(x) \geq 0$ for x in $[0, 1]$.

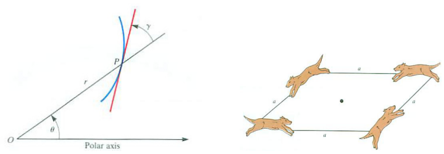
32.[C] Consider the length of the curve $y = x^m$, where m is a rational number. Show that the Fundamental Theorem of Calculus is of aid in computing this length only if $m = 1$ or if m is of the form $1 + 1/n$ for some integer n. *Hint:* Chebyshev proved that $\int x^p(1 + x)^q \, dx$ is elementary for rational numbers p and q only when at least one of p, q and $p + q$ is an integer.

33.[C] If one convex polygon P_1 lies inside another poligon P_2 is the perimeter of P_1 necessarily less than the perimeter of P_2? What if P_1 is not convex?

34.[C] One leaf of the cardioid $r = 1+\sin(\theta)$ is traced as θ increases from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$. Find the highest point on that leaf in polar coordinates.

Exercises 35 and 36 form a unit. **35.**[C] Figure 9.4.8(a) shows the angle between the radius and tangent line to the curve $r = f(\theta)$. Using the fact that $\gamma = \alpha - \theta$ and that $\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$, show that $\tan(\gamma) = \dfrac{r}{r'}$. NOTE: See Exercise 36 for the derivation of $\tan(\gamma)$.

36.[C] The formula $\tan(\gamma) = r/r'$ in Exercise 35 is so simple one would expect a simple geometric explanation. Use the "triangle" in Figure 9.4.5 that we used to obtain the formula for $\frac{ds}{d\theta}$ to show that $\tan(\gamma)$ should be r/r'. NOTE: See Exercise 35.



(a) (b)

Figure 9.4.8: (a) ARTIST: (a) extend the (red) tangent line to the curve so it intersects the polar axis and label the angle made by the tangent to the curve with the polar axis as α

37.[C] Four dogs are chasing each other counterclockwise at the same speed. Initially they are at the four vertices of a square of side a. As they chase each other, each running directly toward the dog in front, they approach the center of the square in spiral paths. How far does each dog travel?

(a) Find the equation of the spiral path each dog follows and use calculus to answer this question.

(b) Answer the question without using calculus.

38.[C] We assumed that a chord $\overline{AB}$ of a smooth curve is a good approximation of the arc $\widehat{AB}$ when B is near to A. Show that the formula we obtained for arc length is consistent with this assumption. That is, if $y = f(x)$, $A = (a, f(a))$, $B = (x, f(x))$, then

$$\frac{\int_a^x \sqrt{1 + f'(t)^2}\, dt}{\sqrt{(x-a)^2 + (f(x) - f(a))^2}}$$

approaches 1 as x approaches a. Assume that $f'(x)$ is continuous. Hint: L'Hôpital's Rule is tempting but does not help. For simplicity, assu

39.[C] In some approaches to ar on a curve the arc length is found f We outline this method in this Exe Let $x = g(t)$, $y = h(t)$ where g and derivatives. Let $a = t_0 < t_1 <$ be a partition of $[a, b]$ into n equa $\Delta t = (b - a)/n$. Let $P_i = (g(t_i), h($ as (x_i, y_i). Then the polygon P_0P_1 in the curve. We assume that as n this polygon, $\sum_{i=1}^{n} |\overline{P_{i-1}P_i}|$ appro the curve from $(g(a), h(a))$ to $(g(b)$

(a) Show that the length o $\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i -$

(b) Show that the sum can be w

$$\sum_{i=1}^{n} \sqrt{(g'(t_i^*))^2 + (h'(t_i^{**}$$

for some t_i^* and t_i^{**} in $[t_{i-1}, t$

(c) Why would you expect th as $n \to \infty$ to be $\int_a^b \sqrt{(g$ Note: This result is typic vanced Calculus, even thoug different.

(d) From (c) deduce that $\sqrt{(g'(t))^2 + h'(t))^2}$.

9.5 The Area of a Surface of Revolution

In this section we develop a formula for expressing the surface area of a solid of revolution as a definite integral. In particular, we will show that the surface area of a sphere is four times the area of a cross section through its center. (See Figure 9.5.1.) This was one of the great discoveries of Archimedes in the third century B.C.

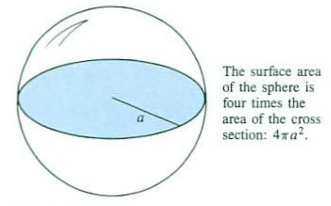
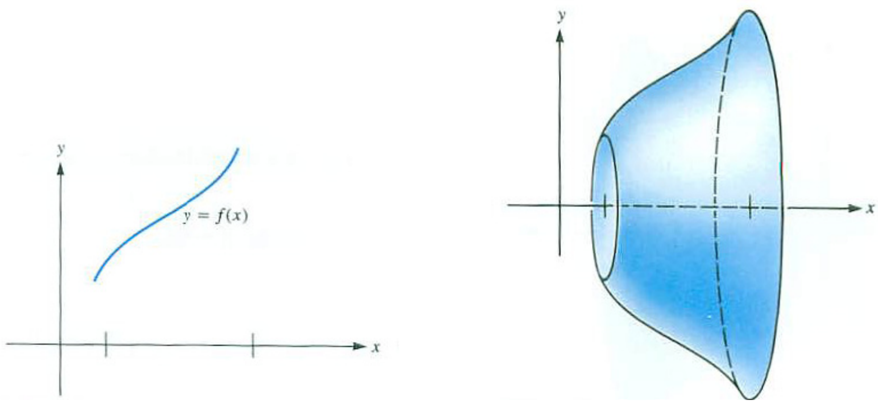


Figure 9.5.1:

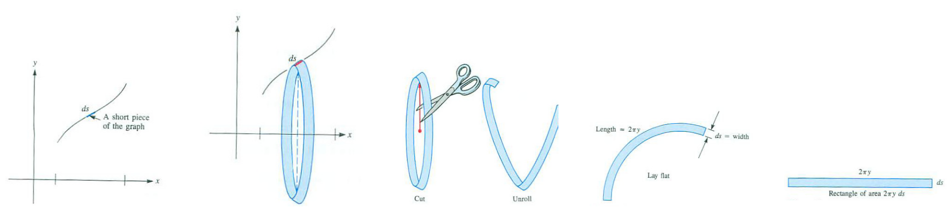
Let $y = f(x)$ have a continuous derivative for x in some interval. Assume that $f(x) \geq 0$ on this interval. When its graph is revolved about the x-axis it sweeps out a surface, as shown in Figures 9.5.2. To develop a definite integral



(a) (b)

Figure 9.5.2:

for this surface area, we use an informal approach.



(a) (b) (c) (d) (e)

Figure 9.5.3:

Consider a very short section of the graph $y = f(x)$. It is almost straight. Let us approximate it by a short line segment of length ds, a very small number. When this small line segment is revolved about the x-axis it sweeps out a narrow band. (See Figures 9.5.3(a) and (b).)

If we can estimate the area of this band, then we will have a local approximation of the surface area. From the local approximation we can set up a definite integral for the entire surface area.

Imagine cutting the band with scissors and laying it flat, as in Figures 9.5.3(c) and (d). It seems reasonable that the area of the flat band in Figure 9.5.3(d) is close to the area of a flat rectangle of length $2\pi y$ and width ds, as in Figure 9.5.3(e). (See Exercises 28 and 29.)

The gives us

$$\text{local approximation of the surface area of one slice} = 2\pi y\ ds.$$

which, in the usual way, leads to the formula

$$\text{Surface area} = \int_{s_0}^{s_1} 2\pi y\ ds. \qquad (9.5.1)$$

where $[s_0, s_1]$ describes the appropriate interval on the "s-axis". Since s is a clumsy parameter, for computations we will use one of the forms for ds to change (9.5.1) into more convenient integrals.

Assume that $y \geq 0$ and that dy/dx is continuous.

Say that the section of the graph $y = f(x)$ that was revolved corresponds to the interval $[a, b]$ on the x-axis, as in Figure 9.5.4. Then

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

and the surface area integral $\int_{s_0}^{s_1} 2\pi y\ ds$ becomes

$$\text{Surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx. \qquad (9.5.2)$$

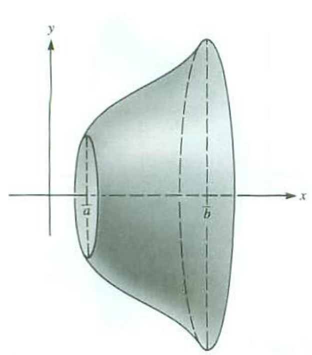


Figure 9.5.4:

EXAMPLE 1 Find the surface area of a sphere of radius a.

SOLUTION The circle of radius a has the equation $x^2 + y^2 = a^2$. The top half has the equation $y = \sqrt{a^2 - x^2}$. The sphere of radius a is formed by revolving this semi-circle about the x-axis. (See Figure 9.5.5.) We have

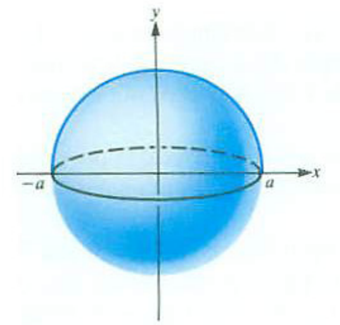


Figure 9.5.5:

$$\text{surface area of sphere} = \int_{-a}^{a} 2\pi y\ ds.$$

Because $dy/dx = -x/\sqrt{a^2 - x^2}$ we find that

$$\begin{aligned} ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2}\ dx \\ &= \sqrt{1 + \frac{x^2}{a^2 - x^2}}\ dx = \sqrt{\frac{a^2}{a^2 - x^2}}\ dx = \frac{a}{\sqrt{a^2 - x^2}}\ dx. \end{aligned}$$

Thus,

$$\begin{aligned} \text{surface area of sphere} &= \int_{-a}^{a} 2\pi y\ ds = \int_{-a}^{a} 2\pi\sqrt{a^2 - x^2}\frac{a}{\sqrt{a^2 - x^2}}\ dx \\ &= \int_{-a}^{a} 2\pi a\ dx = 2\pi a x|_{-a}^{a} = 4\pi a^2. \end{aligned}$$

The surface area of a sphere is 4 times the area of its equatorial cross section. ◇

If the graph is given parametrically, $x = g(t)$, $y = h(t)$, where g and h have continuous derivatives and $h(t) \geq 0$, then it is natural to express the integral $\int_{s_0}^{s_1} 2\pi y\ ds$ as an integral over an interval on the t-axis. If t varies in the interval $[a, b]$, then

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

which leads to

$$\text{Surface area for a parametric curve} = \int_{a}^{b} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt. \qquad (9.5.3)$$

Formula 9.5.2 is just the special case of Formula 9.5.3 when the parameter is x.

As the formulas are stated, they seem to refer only to surfaces obtained by revolving a curve about the x-axis. In fact, they refer to revolution about any line. The factor y in the integrand, $2\pi y\ ds$, is the distance from the typical point on the curve to the axis of revolution. Replace y by R (for *radius*) to free ourselves from coordinate systems. (Use capital R to avoid confusion with polar coordinates.) The simplest way to write the formula for surface area of revolution is then

$$\text{Surface area} = \int_c^d 2\pi R\ ds,$$

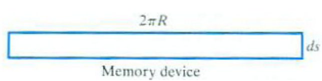


Figure 9.5.6: The key to this section.

where the interval $[c, d]$ refers to the parameter s. However, in practice arc length, s, is seldom a convenient parameter. Instead, x, y, t or θ is used and the interval of integration describes the interval through which the parameter varies.

To remember this formula, think of a narrow circular band of width ds and radius R as having an area close to the area of the rectangle shown in Figure 9.5.6.

EXAMPLE 2 Find the area of the surface obtained by revolving around the y-axis the part of the parabola $y = x^2$ that lies between $x = 1$ and $x = 2$. (See Figure 9.5.7.)

R is found by inspection of a diagram.

SOLUTION The surface area is $\int_a^b 2\pi R\ ds$. Since the curve is described as a function of x, choose x as the parameter. By inspection of Figure 9.5.7, $R = x$. Next, note that

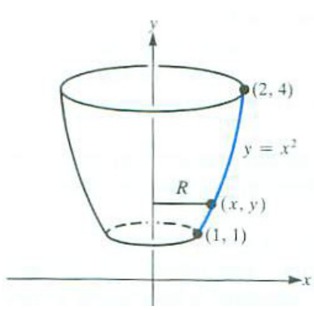


Figure 9.5.7:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \sqrt{1 + 4x^2}\ dx.$$

The surface area is therefore

$$\int_1^2 2\pi x\sqrt{1 + 4x^2}\ dx.$$

To evaluate the integral, use the substitution

$$u = 1 + 4x^2 \qquad du = 8x\ dx.$$

Hence $x\,dx = du/8$. The new limits of integration are $u = 5$ and $u = 17$. Thus

$$\begin{aligned}\text{surface area} &= \int_5^{17} 2\pi\sqrt{u}\frac{du}{8} = \frac{\pi}{4}\int_5^{17}\sqrt{u}\;du \\ &= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_5^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.84649.\end{aligned}$$

◇

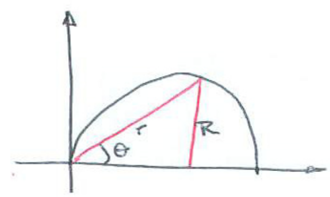


Figure 9.5.8:

EXAMPLE 3 Find the surface area when the curve $r = \cos(\theta)$, θ in $[0, \pi/2]$ is revolved around (a) the x-axis and (b) the y-axis.

SOLUTION The curve is shown in Figure 9.5.8. Note that it is the semicircle with radius $1/2$ and center $(1/2, 0)$. (a) We need to find both R and $ds/d\theta$. First, $R = r\sin(\theta) = \cos(\theta)\sin(\theta)$. And, using the formula for $\frac{ds}{d\theta}$ for a polar curve from Section 9.4 we have

$$\frac{ds}{d\theta} = \sqrt{r(\theta)^2 + r'(\theta)^2} = \sqrt{(\cos(\theta))^2 + (-\sin(\theta))^2} = 1.$$

Then

$$\begin{aligned}\text{surface area} &= \int_0^{\pi/2} 2\pi R\frac{ds}{d\theta}\;d\theta = \int_0^{\pi/2} 2\pi\cos(\theta)\sin(\theta)(1)\;d\theta \\ &= \int_0^{\pi/2} 2\pi\sin(\theta)\cos(\theta)\;d\theta = 2\pi\frac{\sin^2(\theta)}{2}\Big|_0^{\pi/2} = \pi.\end{aligned}$$

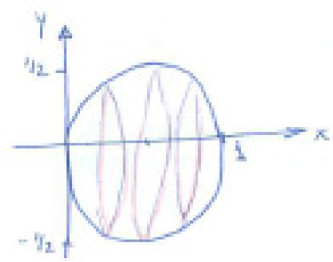


Figure 9.5.9:
Recall the easy way to find

This is expected since this surface of revolution is a sphere of radius $1/2$. See Figure 9.5.9.

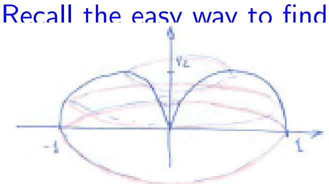


Figure 9.5.10:

$$\begin{aligned}\text{surface area} &= \int_0^{\pi/2} 2\pi R\frac{ds}{d\theta}\;d\theta = \int_0^{\pi/2} 2\pi\cos^2(\theta)(1)\;d\theta \\ &= 2\pi\int_0^{\pi/2}\cos^2(\theta)\;d\theta = 2\pi(\frac{\pi}{4}) = \frac{\pi^2}{2}.\end{aligned}$$

This surface is the top half of a doughnut whose hole has just vanished. See Figure 9.5.10. ◇

Summary

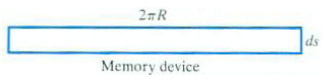


Figure 9.5.11:

This section developed a definite integral for the area of a surface of revolution. It rests on the local estimate of the area swept out by a short segment of length ds revolved around a line L at a distance R from the segment: $2\pi R\ ds$. (See Figure 9.5.11.) We gave an informal argument for this estimate; Exercises 28 and 29 develop it more formally.

EXERCISES for Section 9.5 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 4 set up a definite integral for the area of the indicated surface using the suggested parameter. Show the radius R on a diagram. Do *not* evaluate the definite integrals.

1.[R] The graph of $y = x^3$, x on the interval $[1, 2]$ revolved about the x-axis with parameter x.

2.[R] The graph of $y = x^3$, x on the interval $[1, 2]$ revolved about the line $y = -1$ with parameter x.

3.[R] The graph of $y = x^3$, x on the interval $[1, 2]$ revolved about the y-axis with parameter y.

4.[R] The graph of $y = x^3$, x on the interval $[1, 2]$ revolved about the y-axis with parameter x.

5.[R] Find the area of the surface obtained by rotating about the x-axis that part of the curve $y = e^x$ that lies above $[0, 1]$.

6.[R] Find the area of the surface formed by rotating one arch of the curve $y = \sin(x)$ about the x-axis.

7.[R] One arch of the cycloid given parametrically by the formula $x = \theta - \sin(\theta)$, $y = 1 - \cos(\theta)$ is revolved around the x-axis. Find the area of the surface produced.

8.[R] The curve given parametrically by $x = e^t \cos(t)$, $y = e^t \sin(t)$ $(0 \le t \le \pi/2)$ is revolved around the x-axis. Find the area of the surface produced.

In each of Exercises 9 to 16 find the area of the surface formed by revolving the indicated curve about the indicated axis. Leave the answer as a definite integral, but indicate how it could be evaluated by the Fundamental Theorem of Calculus.

9.[R] $y = 2x^3$ for x in $[0, 1]$; about the x-axis.

10.[R] $y = 1/x$ for x in $[1, 2]$; about the x-axis.

11.[R] $y = x^2$ for x in $[1, 2]$; about the x-axis.

12.[R] $y = x^{4/3}$ for x in $[1, 8]$; about the y-axis.

13.[R] $y = x^{2/3}$ for x in $[1, 8]$; about the line $y = 1$.

14.[R] $y = x^3/6 + 1/(2x)$ for x in $[1, 3]$; about the y-axis.

15.[R] $y = x^3/3 + 1/(4x)$ for x in $[1, 2]$; about the line $y = -1$.

16.[R] $y = \sqrt{1 - x^2}$ for x in $[-1, 1]$; about the line $y = -1$.

17.[M] Consider the smallest tin can that contains a given sphere.[1] (The height and diameter of the tin can equal the diameter of the sphere.)

[1] Archimedes, who obtained the solution about 2200 years ago, considered it his greatest accomplishment. Cicero wrote, about two centuries after Archimedes' death:

> I shall call up from the dust [the ancient equivalent of a blackboard] and his measuring-rod an obscure, insignificant person belonging to the same city [Syracuse], who lived many years after, Archimedes. When I was quaestor I tracked out his grave, which was unknown to the Syracusans (as they totally denied its existence), and found it enclosed all round and covered with brambles and thickets; for I remembered certain doggerel lines inscribed, as I had heard, upon his tomb, which stated that a sphere along with a cylinder had been set up on the top of his grave. Accordingly, after taking a good look around (for there are a great quantity of graves at the Agrigentine Gate), I noticed a small column rising a little above the bushes, on which there was the figure of a sphere and a cylinder. And so I at once said to the Syracusans (I had their leading men with me) that I believed it was the very thing of which I was in search. Slaves were sent in with sickles who cleared the ground of obstacles, and when a passage to the place was opened we approached the pedestal fronting us; the epigram was traceable with about half the lines legible, as the latter portion was worn away. [Cicero, *Tusculan Disputations*, vol. 23, translated by J. E. King, Loef Classical Library, Harvard Univeristy, Cambridge, 1950.]

Archimedes was killed by a Roman soldier in 212 B.C. Cicero was quaestor in 75 B.C.

(a) Compare the volume of the sphere with the volume of the tin can.

(b) Compare the surface area of the sphere with the total surface area of the can.

NOTE: See also Exercise 37.

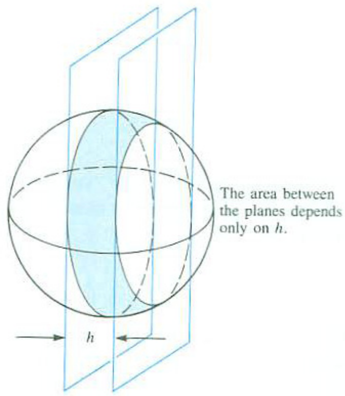


Figure 9.5.12:

18.[M]

(a) Compute the area of the portion of a sphere of radius a that lies between two parallel planes at distances c and $c+h$ from the center of the sphere $(0 \leq c \leq c+h \leq a)$.

(b) The result in (a) depends only on h, not on c. What does this mean geometrically? (See Figure 9.5.12.)

In Exercises 19 and 20 estimate the surface area obtained by revolving the specified arc about the given line. First, find a definite integral for the surface area. Then, use either Simpson's method with six sections or a programmable calculator or computer to approximate the value of the integral.

19.[M] $y = x^{1/4}$, x in $[1,3]$, about the x-axis.

20.[M] $y = x^{1/5}$, x in $[1,3]$, about the line $y = -1$.

Exercises 21 to 24 are concerned with the area of a surface obtained by revolving a curve given in polar coordinates.

21.[M] Show that the area of the surface obtained by revolving the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, around the polar axis is

$$\int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{r^2 + (r')^2}\, d\theta.$$

HINT: Use a local approximation informally.

22.[M] Use Exercise 21 to find the surface area of a sphere of radius a.

23.[M] Find the area of the surface formed by revolving the portion of the curve $r = 1 + \cos(\theta)$ in the first quadrant about (a) the x-axis, (b) the y-axis. HINT: The identity $1 + \cos(\theta) = 2\cos^2(\theta/2)$ may help in (b).

24.[M] The curve $r = \sin(2\theta)$, θ in $[0, \pi/2]$, is revolved around the polar axis. Set up an integral for the surface area.

25.[M] The portion of the curve $x^{2/3} + y^{2/3} = 1$ situated in the first quadrant is revolved around the x-axis. Find the area of the surface produced.

26.[M] Although the Fundamental Theorem of Calculus is of no use in computing the perimeter of the ellipse $x^2/a^2 + y^2/b^2 = 1$, it is useful in computing the surface area of the "football" formed when the ellipse is rotated about one of its axes.

(a) Assuming that $a > b$ and that the ellipse is revolved around the x-axis, find that area.

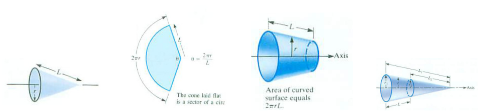
(b) Does your answer give the correct formula for the surface area of a sphere of radius a, $4\pi a^2$? HINT: Let b approach a from the left.

27.[M] The (unbounded) region bounded by $y = 1/x$ and the x-axis and situated to the right of $x = 1$ is revolved around the x-axis.

(a) Show that its volume is finite but its surface area is infinite.

(b) Does this mean that an infinite area can be painted by pouring a finite amount of paint into this solid?

Exercises 28 and 29 obtain the formula for the area of the surface obtained by revolving a line segment about a line that does not meet it. (This area was only estimated in the text.)



(a) (b) (c) (d)

Figure 9.5.13:

28.[M] A right circular cone has slant height L and radius r, as shown in Figure 9.5.13(a). If this cone is cut along a line through its vertex and laid flat, it becomes a sector of a circle of radius L, as shown in Figure 9.5.13(b). By comparing Figure 9.5.13(b) to a complete disk of radius L find the area of the sector and thus the area of the cone in Figure 9.5.13(a).

29.[M] Consider a line segment of length L in the plane which does not meet a certain line in the plane, called the axis. (See Figure 9.5.13(c).) When the line segment is revolved around the axis, it sweeps out a curved surface. Show that the area of this surface equals $2\pi rL$ where r is the distance from the midpoint of the line segment to the axis. The surface in Figure 9.5.3 is called a **frustum of a cone**. Follow these steps:

(a) Complete the cone by extending the frustum as shown in Figure 9.5.13(d). Label the radii and lengths as in that figure. Show that $\frac{r_1}{r_2} = \frac{L_1}{L_2}$, hence $r_1L_2 = r_2L_1$.

(b) Show that the surface area of the frustum is $\pi r_1L_1 - \pi r_2L_2$.

(c) Express L_1 as $L_2 + L$ and, using the result of (a), show that

$$\pi r_1L_1 - \pi r_2L_2 = \pi r_2(L_1 -$$

(d) Show that the surface area of the frustum is $2\pi rL$, where $r = (r_1 + r_2)/2$. NOTE: This justifies our approximation $2\pi R\ ds$.

30.[C] The derivative (with respect to r) of the volume of a sphere is its surface area: $\frac{d}{dr}\left(4\pi r^3/3\right) = 4\pi r^2$. Is this simply a coincidence?

31.[C] Define the **moment of a curve around the x-axis** to be $\int_{s_1}^{s_2} y\ ds$, where s_1 and s_2 refer to the range of the arc length s. The **moment of the curve around the y-axis** is defined as $\int_{s_1}^{s_2} x\ ds$. The **centroid** of the curve, $(\overline{x}, \overline{y})$, is defined by setting

$$\overline{x} = \frac{\int_{s_1}^{s_2} x\ ds}{\text{length of curve}} \qquad \overline{y} = \frac{\int_{s_1}^{s_2} y\ ds}{\text{length of curve}}$$

Find the centroid of the top half of the circle $x^2+y^2=a^2$.

32.[C] Show that the area of the surface obtained by revolving about the x-axis a curve that lies above it is equal to the length of the curve times the distance that the centroid of the curve moves. NOTE: See Exercise 31.

33.[C] Let a be a positive number and $\mathcal{R}$ the region bounded by $y=x^a$, the x-axis, and the line $x=1$.

(a) Show that the centroid of $\mathcal{R}$ is $\left(\frac{a+1}{4a+2}, \left(\frac{a+1}{a+2}\right)^a\right)$.

(b) Show that the centroid of $\mathcal{R}$ lies in $\mathcal{R}$ for all large values of a.

NOTE: It is true that the centroid lies in $\mathcal{R}$ for all positive values of a, but this proof is more difficult.

34.[C] Use Exercise 32 to find the surface area of the doughnut formed by revolving a circle of radius a around a line a distance b from its center, $b \geq a$.

35.[C] Use Exercise 32 to find the area of the curved part of a cone of radius a and height h.

36.[C] For some continuous func nite integral $\int_a^b f(x)\ dx$ depends o the interval $[a,b]$; namely, there is a that

$$\int_a^b f(x)\ dx = g(b -$$

(a) Show that every constant fu (9.5.4).

(b) Prove that if $f(x)$ satisfies (be constant.

NOTE: See Exercise 18.

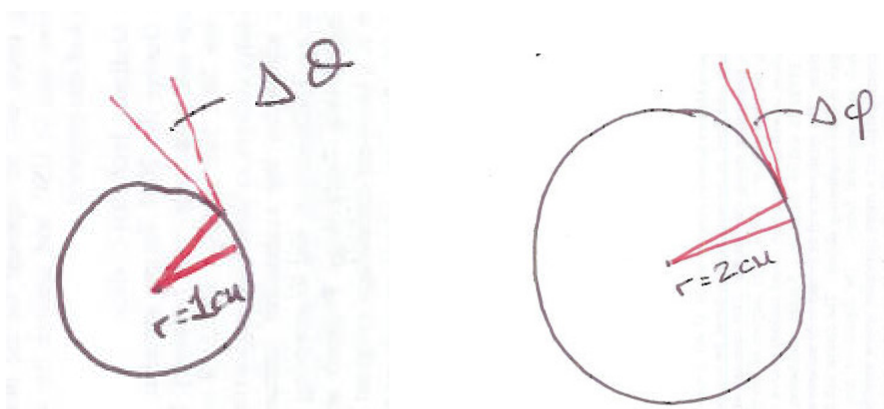
37.[C] The Mercator map discu this chapter preserves angles. A **La equal-area projection** preserves gles. It is made by projecting a s tangent at the equator by rays p torial plane and having one end o joins the north and south poles, as Explain why a Lambert map HINT: See Exercise 17.

9.6 Curvature

In this section we use calculus to obtain a measure of the "curviness" or "curvature" at points on a curve. This concept will be generalized in Section 15.2 in the study of motion along a curved path in space.

Introduction

Imagine a bug crawling around a circle of radius one centimeter, as in Figure 9.6.1(a). As it walks a small distance, say 0.1 cm, it notices that its direction, measured by angle θ, changes. Another bug, walks around a larger circle, as in Figure 9.6.1(b). Whenever it goes 0.1 cm, its direction, measured by angle ϕ, changes by much less. The first bug feels that his circle is curvier than the circle of the second bug. We will provide a measure of "curviness" or **curvature**. A straight line will have "zero curvature" everywhere. A circle of radius a will turn out to have curvature $1/a$ everywhere. For other curves, the curvature varies from point to point.



(a) (b)

Figure 9.6.1: The circle in (b) has twice the radius as the circle in (a). But, the change in $\Delta\phi$ in (b) is half that in (a).

Definition of Curvature

"Curvature" measures how rapidly the direction changes as we move a small distance along a curve. We have a way of assigning a numerical value to direction, namely, the angle of the tangent line. The *rate of change of this angle with respect to arc length* will be our measure of curvature.

DEFINITION *(Curvature)* Assume that a curve is given parametrically, with the parameter of the typical point P being s, the distance along the curve from a fixed P_0 to P. Let ϕ be the angle between the tangent line at P and the positive part of the x-axis. The **curvature** κ at P is the absolute value of the derivative, $\frac{d\phi}{ds}$:

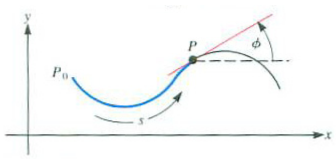


Figure 9.6.2:

κ is the Greek letter "kappa".

$$\text{curvature} = \kappa = \left|\frac{d\phi}{ds}\right|$$

whenever the derivative exists. (See Figure 9.6.2.)

Observe that a straight line has zero curvature everywhere, since ϕ is constant.

The next theorem shows that curvature of a small circle is large and the curvature of a large circle is small, in agreement with the bugs' experience.

Theorem. *(Curvature of Circles) For a circle of radius a, the curvature $\left|\frac{d\phi}{ds}\right|$ is constant and equals $1/a$, the reciprocal of the radius.*

Proof

It is necessary to express ϕ as a function of arc length s on a circle of radius a. Refer to Figure 9.6.3. Arc length s is measured counterclockwise from the point P_0 on the x-axis. Then $\phi = \frac{\pi}{2} + \theta$, as Figure 9.6.3 shows. By definition of radian measure, $s = a\theta$, so that $\theta = s/a$. We can solve for ϕ, $\phi = \frac{\pi}{2} + \frac{s}{a}$. Differentiating with respect to arc length yields:

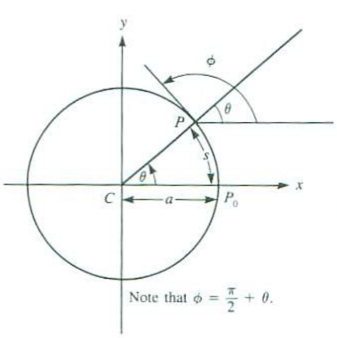


Figure 9.6.3:

$$\frac{d\phi}{ds} = \frac{1}{a},$$

as claimed. •

Computing Curvature

When a curve is given in the form $y = f(x)$, the curvature can be expressed in terms of the first and second derivatives, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Theorem. *(Curvature of $y = f(x)$) Let arc length s be measured along the curve $y = f(x)$ from a fixed point P_0. Assume that x increases as s increases and that y' and y'' are continuous. Then*

The curvature of $y = f(x)$.

$$\text{curvature} = \kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}.$$

Proof

The Chain Rule, $\frac{d\phi}{dx} = \frac{d\phi}{ds}\frac{ds}{dx}$, implies

$$\frac{d\phi}{ds} = \frac{\frac{d\phi}{dx}}{\frac{ds}{dx}}.$$

As was shown in Section 9.3,

$$\frac{ds}{dx} = \left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2}.$$

All that remains is to express $\frac{d\phi}{dx}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Note that in Figure 9.6.4,

$$\frac{dy}{dx} = \text{slope of tangent line to the curve} = \tan(\phi). \qquad (9.6.1)$$

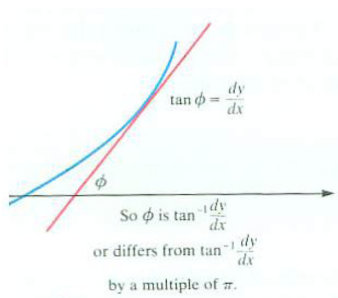


Figure 9.6.4:

We find $\frac{d\phi}{dx}$ by differentiating both sides of (9.6.1) with respect to x, that is, both sides of the equation $\frac{dy}{dx} = \tan(\phi)$. Thus

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan(\phi)) = \sec^2(\phi)\cdot\frac{d\phi}{dx} = \left(1+\tan^2(\phi)\right)\frac{d\phi}{dx} = \left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{d\phi}{dx}.$$

Solving for $d\phi/dx$, we get

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}.$$

Consequently,

$$\frac{d\phi}{ds} = \frac{\frac{d\phi}{dx}}{\frac{ds}{dx}} = \frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}},$$

and the theorem is proved. •

WARNING *(Geometry of the Curvature)* One might have expected the curvature to depend only on the second derivative, $\frac{d^2y}{dx^2}$, since it records the rate at which the slope changes. This expectation is correct only when $\frac{dy}{dx} = 0$, that is, at critical points in the graph of $y = f(x)$. (See also Exercise 28.)

EXAMPLE 1 Find the curvature at a point (x, y) on the curve $y = x^2$.
SOLUTION In this case $\frac{dy}{dx} = 2x$ and $\frac{d^2y}{dx^2} = 2$. The curvature at (x, y) is

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}} = \frac{2}{(1 + (2x)^2)^{3/2}}.$$

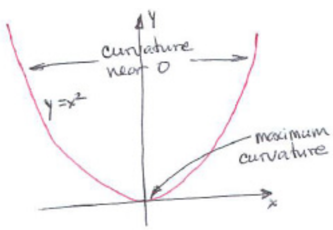


Figure 9.6.5:

The maximum curvature occurs when $x = 0$. The curvatures at (x, x^2) and at $(-x, x^2)$ are equal. As $|x|$ increases, the curve becomes straighter and the curvature approaches 0. (See Figure 9.6.5.) ◇

Curvature of a Parameterized Curve

Theorem 9.6 applies also to curves given parametrically.

Theorem 9.6 tells how to find the curvature if y is given as a function of x. But it holds as well when the curve is described parametrically, where x and y are functions of some parameter such as t or θ. Just use the fact that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}. \tag{9.6.2}$$

Both equations in (9.6.2) are special cases of

$$\frac{df}{dx} = \frac{\frac{df}{dt}}{\frac{dx}{dt}}.$$

And this equation is just the Chain Rule in disguise,

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}.$$

In the first equation in (9.6.2), the function f is y; in the second equation, f is $\frac{dy}{dx}$. Example 2 illustrates the procedure.

EXAMPLE 2 The cycloid determined by a wheel of radius 1 has the parametric equations

$$x = \theta - \sin(\theta) \qquad \text{and} \qquad y = 1 - \cos(\theta),$$

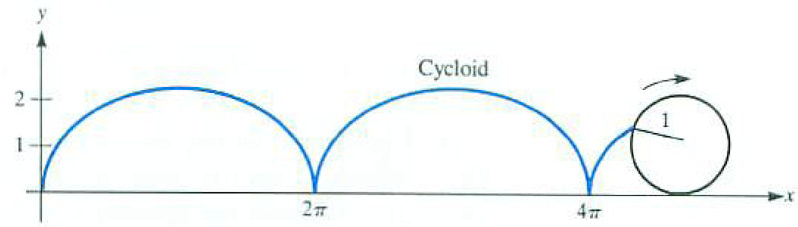


Figure 9.6.6:

as shown in Figure 9.6.6. Find the curvature at a typical point on this curve.
SOLUTION First find $\frac{dy}{dx}$ in terms of θ. We have

$$\frac{dx}{d\theta} = 1 - \cos(\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = \sin(\theta).$$

Thus

$$\frac{dy}{dx} = \frac{\sin(\theta)}{1 - \cos(\theta)}.$$

Similar direct calculations show that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left(\frac{\sin(\theta)}{1-\cos(\theta)}\right)}{1 - \cos(\theta)} = \frac{-1}{(1 - \cos(\theta))^2}.$$

The parts of the cycloid near the x-axis are nearly vertical. See Exercise 29.

Thus the curvature is

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}} = \frac{\left|\frac{-1}{(1-\cos(\theta))^2}\right|}{\left(\frac{2}{1-\cos(\theta)}\right)^{3/2}} = \frac{1}{2^{3/2}\sqrt{1 - \cos(\theta)}}.$$

Since $y = 1 - \cos(\theta)$ and $2^{3/2} = \sqrt{8}$, the curvature equals $1/\sqrt{8y}$. ⋄

Radius of Curvature

As Theorem 9.6 shows, a circle with curvature κ has radius $1/\kappa$. This suggests the following definition.

A large radius of curvature implies a small curvature.

DEFINITION *(Radius of Curvature)* The **radius of curvature** of a curve at a point is the reciprocal of the curvature:

$$\text{radius of curvature} = \frac{1}{\text{curvature}} = \frac{1}{\kappa}.$$

As can be easily checked, the radius of curvature of a circle of radius a is, fortunately, a.

The cycloid in Example 2 has radius of curvature at the point (x, y) equal to $\sqrt{8y}$. The higher the point on the curve, the straighter the curve.

The line through a point P as a curve that looks most like the curve near P is the tangent line. The circle through P that looks most like the curve near P has the same slope at P as the curve and a radius equal to the radius of curvature at P. It is called the **osculating circle**, from the Latin "osculum = kiss." The tangent line is never called the "osculating line".

The Osculating Circle

At a given point P on a curve, the **osculating circle** at P is defined to be that circle which (a) passes through P, (b) has the same slope at P as the curve does, and (c) has the same curvature there.

For instance, consider the parabola $y = x^2$ of Example 1. When $x = 1$, the curvature is $2/5^{3/2}$ and the radius of curvature is $5^{3/2}/2 \approx 5.59017$. The osculating circle at $(1, 1)$ is shown in Figure 9.6.7.

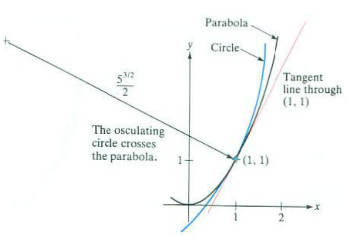


Figure 9.6.7:

Observe that the osculating circle in Figure 9.6.7 *crosses the parabola* as it passes through the point $(1, 1)$. Although this may be surprising, a little reflection will show why it is to be expected.

Think of driving along the parabola $y = x^2$. If you start at $(1, 1)$ and drive up along the parabola, the curvature diminishes. It is smaller than that of the circle of curvature at $(1, 1)$. Hence you would be turning your steering wheel to the left and would be traveling *outside* the osculating circle at $(1, 1)$. On the other hand, if you start at $(1, 1)$ and move toward the origin (to the left) on the parabola, the curvature increases and is greater than that of the osculating circle at $(1, 1)$, so you would be driving *inside* the osculating circle at $(1, 1)$. This informal argument shows why the osculating circle crosses the curve in general. In the case of $y = x^2$, the only osculating circle that does *not* cross the curve at its point of tangency is the one that is tangent at $(0, 0)$, where the curvature is a maximum.

Summary

We defined the curvature κ of a curve as the absolute value of the rate at which the angle between the tangent line and the x-axis changes as a function of arc length; curvature equals $\left|\frac{d\phi}{ds}\right|$. If the curve is the graph of $y = f(x)$, then

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}.$$

If the curve is given in terms of a parameter t then compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ with the aid of the relation

Equation (9.6.3) is our old friend, the Chain Rule; just clear the denominator.

$$\frac{d(\ )}{dx} = \frac{\frac{d(\ )}{dt}}{\frac{dx}{dt}}, \tag{9.6.3}$$

the empty parentheses enclosing first y, then $\frac{dy}{dx}$.

Radius of curvature is the reciprocal of curvature.

EXERCISES for Section 9.6 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 6 find the curvature and radius of curvature of the specified curve at the given point.

1.[R] $y = x^2$ at $(1, 1)$

2.[R] $y = \cos(x)$ at $(0, 1)$

3.[R] $y = e^{-x}$ at $(1, 1/e)$

4.[R] $y = \ln(x)$ at $(e, 1)$

5.[R] $y = \tan(x)$ at $(\frac{\pi}{4}, 1)$

6.[R] $y = \sec(2x)$ at $(\frac{\pi}{6}, 2)$

In Exercises 7 to 10 find the curvature of the given curves for the given value of the parameter.

7.[R] $\begin{cases} x = 2\cos(3t) \\ y = 2\sin(3t) \end{cases}$ at $t = 0$

8.[R] $\begin{cases} x = 1 + t^2 \\ y = t^3 + t^4 \end{cases}$ at $t = 2$

9.[R] $\begin{cases} x = e^{-t}\cos(t) \\ y = e^{-t}\sin(t) \end{cases}$ at $t = \frac{\pi}{6}$

10.[R] $\begin{cases} x = \cos^3(\theta) \\ y = \sin^3(\theta) \end{cases}$ at $\theta = \frac{\pi}{3}$

11.[R]

(a) Compute the curvature and radius of curvature for the curve $y = (e^x + e^{-x})/2$.

(b) Show that the radius of curvature at (x, y) is y^2.

12.[R] Find the radius of curvature along the curve $y = \sqrt{a^2 - x^2}$, where a is a constant. (Since the curve is part of a circle of radius a, the answer should be a.)

13.[R] For what value of x is the radius of curvature of $y = e^x$ smallest?
Hint: How does one find the minimum of a function?

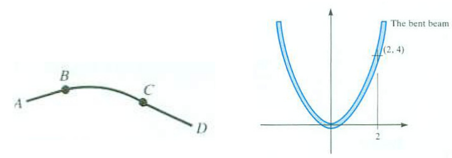
14.[R] For what value of x is the radius of curvature of $y = x^2$ smallest?

15.[M]

(a) Show that at a point where a curve has its tangent parallel to the x-axis its curvature is simply the absolute value of the second derivative d^2y/dx^2.

(b) Show that the curvature is never larger than the absolute value of d^2y/dx^2.

16.[M] An engineer lays out a railroad track as indicated in Figure 9.6.8(a). BC is part of a circle; AB and CD are straight and tangent to the circle. After the first train runs over the track, the engineer is fired because the curvature is not a continuous function. Why should the curvature be continuous?



(a) (b)

Figure 9.6.8:

17.[M] Railroad curves are banked to reduce wear on the rails and flanges. The greater the radius of curvature, the less the curve must be banked. The best bank angle A satisfies the equation $\tan(A) = v^2/(32R)$, where v is speed in feet per second and R is radius of curvature in feet. A train travels in the elliptical track

$$\frac{x^2}{1000^2} + \frac{y^2}{500^2} = 1$$

at 60 miles per hour. Find the best angle A at the points $(1000, 0)$ and $(0, 500)$. NOTE: x and y are measured in feet; 60 mph=88 fps.

18.[M] The flexure formula in the theory of beams asserts that the bending moment M required to bend a beam is proportional to the desired curvature, $M = c/R$, where c is a constant depending on the beam and R is the radius of curvature. A beam is bent to form the parabola $y = x^2$. What is the ratio between the moments required at (a) at $(0,0)$ and (b) at $(2,4)$? (See Figure 9.6.8(b).)

Exercises 19 to 21 are related.

19.[M] Find the radius of curvature at a typical point on the curve whose parametric equations are

$$x = a\cos\theta, \qquad y = b\sin\theta.$$

20.[M]

(a) Show, by eliminating θ, that the curve in Exercise 19 is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(b) What is the radius of curvature of this ellipse at $(a,0)$? at $(0,b)$?

21.[M] An ellipse has a major diameter of length 6 and a minor diameter of length 4. Draw the circles that most closely approximate this ellipse at the four points that lie at the extremities of its diameters. (See Exercises 19 and 20.)

In each of Exercises 22 to 24 a curve is given in polar coordinates. To find its curvature write it in rectangular coordinates with parameter θ, using the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

22.[M] Find the curvature of $r = a\cos(\theta)$.

23.[M] Show that at the point (r,θ) the cardioid $r = 1 + \cos(\theta)$ has curvature $3\sqrt{2}/(4\sqrt{r})$.

24.[M] Find the curvature of $r = \cos(2\theta)$.

25.[M] If, on a curve, $dy/dx = y^3$, express the curvature in terms of y.

26.[M] As is shown in physics, the larger the radius of curvature of a turn, the faster a given car can travel around that turn. The required radius of curvature is proportional to the square of the maximum speed. Or, conversely, the maximum speed around a turn is proportional to the square root of the radius of curvature. If a car moving on the path $y = x^3$ (x and y measured in miles) can go 30 miles per hour at $(1,1)$ without sliding off, how fast can it go at $(2,8)$?

27.[M] Find the local extrema of the curvature of

(a) $y = x + e^x$

(b) $y = e^x$

(c) $y = \sin(x)$

(d) $y = x^3$

28.[M] Sam says, "I don't like the definition of curvature. It should be the rate at which the slope changes as a function of x. That is $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, which is the second derivative, $\frac{d^2y}{dx^2}$." Give an example of a curve which would have constant curvature according to Sam's definition, but whose changing curvature is obvious to the naked eye.

29.[M] In Example 2 some of the steps were omitted in finding that the cycloid given by $x = \theta - \sin(\theta)$, $y = 1 - \cos(\theta)$ has curvature $\kappa = 1/(2^{3/2}\sqrt{1-\cos(\theta)}) = 1/\sqrt{8y}$. In this exercise you are asked to show all steps in this calculation.

(a) Verify that $\dfrac{dy}{dx} = \dfrac{\sin(\theta)}{1-\cos(\theta)}$.

(b) Show that $\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right) = \dfrac{-1}{1-\cos(\theta)}$

(c) Verify that $\dfrac{d^2y}{dx^2} = \dfrac{-1}{(1-\cos(\theta))^2}$.

(d) Show that $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{2}{1-\cos(\theta)}$.

(e) Compute the curvature, κ, in terms of θ.

(f) Express the curvature found in (e) in terms of x and y.

(g) At which points on the cycloid is the curvature largest?

(h) At which points on the cycloid is the curvature smallest?

30.[M] Assume that g and h are functions with continuous second derivatives. In addition, assume as we move along the parameterized curve $x = g(t)$, $y = h(t)$, the arc length s from a point P_0 increases as t increases. Show that

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

NOTE: Newton's dot notation for derivatives shortens the formula: $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $\dot{y} = \frac{dy}{dt}$, and $\ddot{y} = \frac{d^2y}{dt^2}$.

31.[M] Use the result of Exercise 30 to find the curvature of the cycloid of Example 2. NOTE: $x = \theta - \sin(\theta)$, $y = 1 - \cos(\theta)$

32.[C] (Contributed by G.D. Chakerian) If a planar curve has a constant radius of curvature must it be part of a circle? That the answer is "yes" is important in experiments conducted with a cyclotron: Physical assumptions imply that the path of an electron entering a uniform magnetic field at right angles to the field has constant curvature. Show that it follows that the path is part of a circle.

(a) Show that $\frac{ds}{d\phi} = R$, the radius of curvature.

(b) Show that $\frac{dx}{d\phi} = R\cos(\phi)$.

(c) Show that $\frac{dy}{d\phi} = R\sin(\phi)$.

(d) With the aid of (b) and (c), the curvature involving x an

HINT: For (b) and (c) draw the l hypotenuse is like a short piece of a curve and whose legs are parallel t think about antiderivatives. NOT why the radius of curvature is co the mathematicians to show that t a circle.

33.[C] At the top of the cycloi radius of curvature is twice the dia circle. What would you have gu curvature to be at this point? Why diameter of the wheel, since the wh is rotating about its point of conta

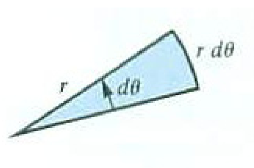
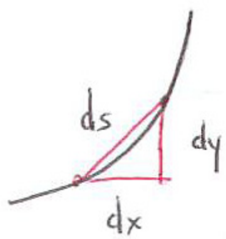
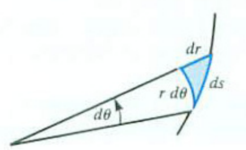
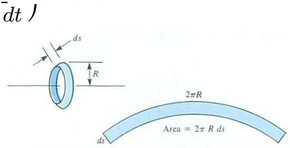
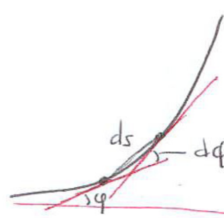
34.[C] A smooth convex curve ha

(a) Show that the average of its tion of arc length, is $2\pi/L$.

(b) Check that the formula in circle of radius a.

9.S Chapter Summary

This chapter deals mostly with curves described in polar coordinates and curves given parametrically. The following table is a list of reminders for most of the ideas in the chapter.

Concept	*Memory Aid*	*Comment*
Area $= \int_\alpha^\beta \frac{r^2}{2}\, d\theta$		The narrow sector resembles a triangle of base $r\ d\theta$ and height r, so $dA = \frac{1}{2}(r\ d\theta)(r) = \frac{1}{2}r^2\ d\theta$.
Arc length $= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$ Arc length $= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$		A short part of the curve is almost straight, suggesting $(ds)^2 = (dx)^2 + (dy)^2$.
Arc length $= \int_\alpha^\beta \sqrt{r^2 + (r')^2}\ d\theta$ $= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right.}$ Speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right.}$ $= \sqrt{\left(r\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)}$		The shaded area with two curved sides looks like a right triangle, suggesting $(ds)^2 = (rd\theta)^2 + (dr)^2$.
Area of surface of revolution $= \int_a^b 2\pi R\ ds$		
Curvature $= \kappa = \left\lvert\frac{d\phi}{ds}\right\rvert$		Using the chain rule to write $\left\lvert\frac{d\phi}{ds}\right\rvert$ as $\left\lvert\frac{(d\phi/dx)}{(ds/dx)}\right\rvert$ one gets the formula $\kappa = \frac{\lvert y''\rvert}{(1+(y')^2)^{3/2}}$

If a curve is given parametrically, its curvature can be found by replacing $\frac{dy}{dx}$ by $\frac{dy/dt}{dx/dt}$, and, similarly, $\frac{d^2y}{dx^2} = \frac{\frac{d}{dx}}{\left(\frac{dy}{dx}\right)}$ by $\frac{\frac{d}{dy}\left(\frac{dy}{dx}\right)}{dx/dt}$.

Section 15.2 defines curvature of a curve in space, using vectors. It is consistent with the definition given here for curves that happen to lie in a plane.

EXERCISES for 9.S *Key:* R–routine, M–moderate, C–challenging

1.[R] When driving along a curvy road, which is more important in avoiding car sickness, $d\phi/ds$ or $d\phi/dt$, where t is time.

2.[R] Some definite integrals can be evaluated by interpretting them as the area of an appropriate region. Consider $\int_0^{\pi/2} \cos^2(\theta)\ d\theta$.

(a) Evaluate $\int_0^{\pi/2} \cos^2(\theta)\ d\theta$ by identifying it as the area of an appropriate region.

(b) Evaluate $\int_0^{\pi/2} \cos^2(\theta)\ d\theta$ with the use of a double angle formula.

(c) Repeat (a) and (b) for $\int_0^{\pi} \sin^2(\theta)\ d\theta$.

(d) Repeat (a) and (b) for $\int_{\pi}^{2\pi} \sin^2(\theta)\ d\theta$.

3.[R] The solution to Example 3 (Section 9.2) requires the evaluation of the definite integrals $\int_0^{\pi/2} \cos^2(\theta)\ d\theta$ and $\int_0^{\pi} (1+\cos(\theta))^2\ d\theta$. Evaluate these definite integrals making use of the ideas in Exercise 2 as much as possible.

4.[M] A physics midterm includes the following information: For $r = \sqrt{x^2+y^2}$ and y constant,

(a) $\displaystyle\int \frac{dx}{r} = \ln(x+r)$, (b) $\displaystyle\int \frac{x\ dx}{r} = r$, (c) $\displaystyle\int \frac{dx}{r^3} = \frac{x}{y^2 r}$.

Show by differentiating that these equations are correct.

5.[M] (Contributed by Jeff Lichtman.) Let f and g be two continuous functions such that $f(x) \geq g(x) \geq 0$ for x in $[0,1]$. Let R be the region under $y = f(x)$ and above $[0,1]$; let R^* be the region under $y = g(x)$ and above $[0,1]$.

(a) Do you think the center of mass of R is at least as high as the center of mass of R^*? (Give your opinion, without any supporting calculations.)

(b) Let $g(x) = x$. Define $f(x)$ to be $\frac{1}{3}$ for $0 \leq x \leq \frac{1}{3}$ and let $f(x)$ be x if $\frac{1}{3} \leq x \leq 1$. (Note that f is continuous.) Find $\bar{y}$ for R and also for R^*. (Which is larger?)

(c) Let a be a constant, $0 \leq a \leq 1$. Let $f(x) = a$ for $0 \leq x \leq a$, and let $f(x) = x$ for $a \leq x \leq 1$. Find $\bar{y}$ for R.

(d) Show that the number a for which $\bar{y}$ defined in (c) is a minimum is a root of the equation $x^3 + 3x - 1 = 0$.

(e) Show that the equation in (d) has only one real root q.

(f) Find q to four decimal places.

(g) Show that $\bar{y} = q$

Exercises 6 and 7 require an integral version of the Cauchy-Schwarz inequality (see Exercise 29):

$$\int_0^{2\pi} f(\theta)g(\theta)\ d\theta \leq \left(\int_0^{2\pi} f(\theta)^2\ d\theta\right)^{1/2} \left(\int_0^{2\pi} g(\theta)^2\ d\theta\right)^{1/2}.$$

6.[C] Let P be a point inside a region in the plane bounded by a smooth convex curve. ("Smooth" means it has a continuously defined tangent line.) Place the pole of a polar coordinate system at P. Let $d(\theta)$ be the length of the chord of angle θ through P. Show that $\int_0^{2\pi} d(\theta)^2\ d\theta \leq 8A$, where A is the area of the region.

7.[C] Show that if $\int_0^{2\pi} d(\theta)^2\ d\theta = 8A$ then P is the midpoint of each chord through P.

8.[C] Let $r = f(\theta)$ describe a convex curve surrounding the origin.

(a) Show that $\int_0^{2\pi} r\ d\theta \leq$ arc length of the boundary.

(b) Show that if equality holds in (a), the curve is a circle.

9.[C] Let $r(\theta)$, $0 \leq \theta \leq 2\pi$, describe a closed convex curve of length L.

(a) Show that the average value of $r(\theta)$, as a function of θ, is at most $L/(2\pi)$.

(b) Show that the if average is $L/(2\pi)$, then the curve is a circle.

10.[C]

Sam: I've discovered an easy formula for the length of a closed curve that encloses the origin.

Jane: Well?

Sam: First of all, $\int_0^{2\pi} \sqrt{r^2 + (r')^2}\ d\theta$ is obviously greater than or equal to $\int_0^{2\pi} \mathbf{r}\ d\theta$.

Jane: I'll give you this much, because $(r')^2$ is never negative.

Sam: Now, if a and b are not negative, $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.

Jane: Why?

Sam: Just square both sides. So $\sqrt{r^2 + (r')^2} \leq \sqrt{r^2} + \sqrt{(r')^2} = r + r'$.

Jane: Looks all right.

Sam: Thus

$$\int_0^{2\pi} \sqrt{r^2 + (r')^2}\ d\theta \leq \int_0^{2\pi} (r+r')\ d\theta = \int_0^{2\pi} r\ d\theta + \int_0^{2\pi} r'\ d\theta$$

But $\int_0^{2\pi} r'\ d\theta$ equals $r(2\pi) - r(0)$, which is 0. So, putting all this together, I get

$$\int_0^{2\pi} r\ d\theta \leq \int_0^{2\pi} \sqrt{r^2 + (r')^2}\ d\theta \leq \int_0^{2\pi} r\ d\theta.$$

So the arc length is simply $\int_0^{2\pi} r\ d\theta$.

Jane: That couldn't be right. If it were, it would be an Exercise.

Sam: They like to keep a few things secret to surprise us on a mid-term.

Who is right, Sam or Jane? Explain.

Skill Drill: Derivatives

In Exercises 11 and 12 a, b, c, m, and p are constants. In each case verify that the derivative of the first function is the second function.

11.[R] $\frac{1}{\sqrt{c}} \arcsin\left(\frac{cx-b}{\sqrt{b^2+ac}}\right)$; $\sqrt{\frac{c}{a+2bx-cx^2}}$.

12.[R] $\frac{1}{c}\sqrt{a+2bx+cx^2} - \frac{b}{\sqrt{c}} \ln\left(b + cx + \sqrt{c}\sqrt{a+2bx+cx^2}\right)$; $\frac{x}{a+bx+cx^2}$ (assume c is positive).

In Exercises 13 and 14, L is the length of a smooth curve C and P is a point within the region A bounded by C.

13.[M]

(a) Show that the average distance from P to points on the curve, averaged with respect to arc length is greater than or equal to $2A/L$.

(b) Give an example when equalify holds.

14.[M]

(a) Show that the average distance from P to points on the curve, averaged with respect to the polar angle is greater than or equal to $L/(2\pi)$.

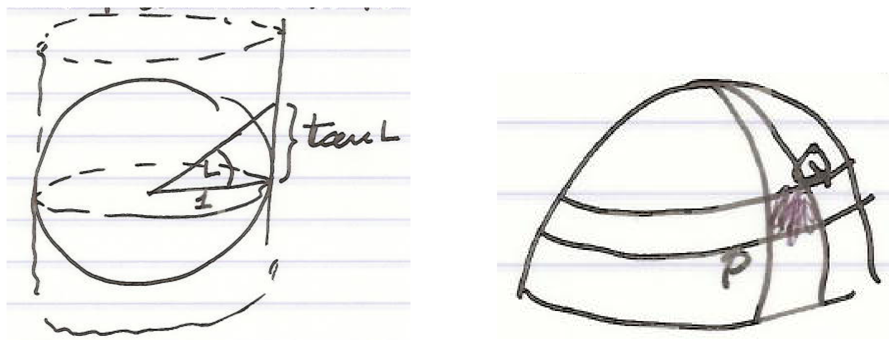
(b) Give an example when equalify holds.

(See also Exercise 24 in Section 9.4.)

Calculus is Everywhere # 11
The Mercator Map

A web search for "map projection" leads to detailed information about this and other projections. The US Geological Society has some particularly good resources.

One way to make a map of a sphere is to wrap a paper cylinder around the sphere and project points on the sphere onto the cylinder by rays from the center of the sphere. This **central cylindrical projection** is illustrated in Figure C.11.1(a).



(a) (b)

Figure C.11.1:

Points at latitude L project onto points at height $\tan(L)$ from the equatorial plane.

A **meridean** is a great circle passing through the north and south poles. It corresponds to a fixed longitude. A short segment on a meridian at latitude L of length dL projects onto the cylinder in a segment of length approximately $d(\tan(L)) = \sec(L)^2\ dL$. This tells us that the map magnifies short vertical segments at latitude L by the factor $\sec^2(L)$.

Points on the sphere at latitude L form a circle of radius $\cos(L)$. The image of this circle on the cylinder is a circle of radius 1. That means the projection magnifies horizontal distances at latitude L by the factor $\sec(L)$.

Consider the effect on a small "almost rectangular" patch bordered by two meridians and two latitude lines. The patch is shaded in Figure C.11.1(b). The vertical edges are magnified by $\sec^2(L)$, but the horizontal edges by only $\sec(L)$. The image on the cylinder will not resemble the patch, for it is stretched more vertically than horizontally.

The path a ship sailing from P to Q makes a certain angle with the latitude line through P. The map just described distorts that angle.

The ship's caption would prefer a map without such a distortion, one that preserves direction. That way, to chart a voyage from point A to point B on the sphere of the Earth at a fixed compass heading, he would simply draw a straight line from A to B on the map to determine the compass setting.

Gerhardus Mercator, in 1569, designed a map that does not distort small patches hence preserves direction. He figured that since the horizontal magnification factor is $\sec(L)$, the vertical magnification should also be $\sec(L)$, not $\sec^2(L)$.

This condition can be stated in the language of calculus. Let y be the height on the map that represents latitude L_0. Then Δy should be approximately $\sec(L)\Delta L$. Taking the limit of $\Delta y/\Delta L$ and ΔL approaches 0, we see that $dy/dL = \sec(L)$. Thus

$$y = \int_0^{L_0} \sec(L)\; dL. \qquad \text{(C.11.1)}$$

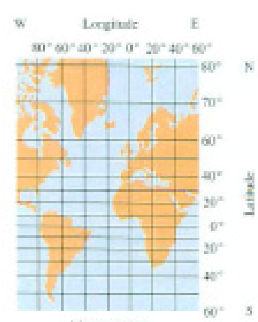
Mercator, working a century before the invention of calculus, did not have the concept of the integral or the Fundamental Theorem of Calculus. Instead, he had to break the interval $[0, L_0]$ into several short sections of length ΔL, compute $(\sec(L))\Delta L$ for each one, and sum these numbers to estimate y in (C.11.1).

We, coming after Newton and Leibniz, can write

$$y = \int_0^{L_0} \sec(L)\, dL = \ln|\sec(L) + \tan(L)|\,\Big|_0^{L_0} = \ln(\sec(L_0)+\tan(L_0)) \qquad \text{for } 0 \le L_0 \le \pi/2.$$

In 1645, Henry Bond conjectured that, on the basis of numerical evidence, $\int_0^\alpha \sec(\theta)\, d\theta = \ln(\tan(\alpha/2+\pi/4))$ but offered no proof. In 1666, Nicolaus Mercator (no relation to Gerhardus) offered the royalties on one of his inventions to the mathematician who could prove Bond's conjecture was right. Within two years James Gregory provided the missing proof.

Figure 11 shows a Mercator map. Such a map, though it preserves angles, greatly distorts areas: Greenland looks bigger than South America even though it is only one eighth its size. The first map we described distorts areas even more than does a Mercator map.



EXERCISES

1.[R] Draw a clear diagram showing why segments at latitude L are magnified vertically by the factor $\sec(L)$.

2.[R] Explain why a short arc of length dL in Figure C.11.1(a) projects onto a short interval of length approximately $\sec^2(L)\; dL$.

3.[R] On a Mercator map, what is the ratio between the distance between the lines representing latitudes 60° and 50° to the distance between the lines representing latitudes 40° amd 30°?

4.[M] What magnifying effect does a Mercator map have on areas?