Chapter 1

Pre-Calculus Review

This chapter reviews precalculus concepts that will be needed often, starting in Chapter 2.

Calculus is the study of functions. To understand the concepts introduced in this text, it is important to have a solid understanding of functions.

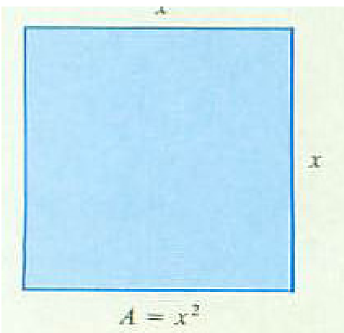
We begin this chapter with a review in Section 1.1 of the terminology for functions that will be used for the remainder of this book. In Section 1.2 fundamental types of functions are reviewed: power functions, exponentials, logarithms, and the trigonometric functions. Section 1.3 describes how functions can be combined to create new functions.

The final two sections of the chapter provide a review of two important topics that will be used in many times in our study of calculus: geometric series in Section 1.4 and logarithms in Section 1.5.

1.1 Functions

This section reviews several ideas related to functions: piecewise-defined function, one-to-one function, inverse function, and increasing or decreasing functions.

Definition of a Function



The area A of a square depends on the length of its side x and is given by the formula $A = x^2$. (See Figure 1.1.1.)

Similarly, the distance s (in feet) that a freely falling object drops in the first t seconds is described by the formula $s = 16t^2$. Each choice of t determines a specific value for s. For instance, when $t = 3$ seconds, $s = 16 \cdot 3^2 = 144$ feet.

Both of these formulas illustrate the notion of a function.

> **DEFINITION** *(Function.)* Let X and Y be sets. A **function** from X to Y is a rule (or method) for assigning one (and only one) element in Y to each element in X.

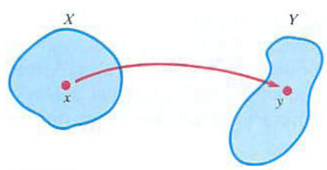


Figure 1.1.2:

The notion of a function is illustrated in Figure 1.1.2, where the element y in Y is assigned to the element x in X. Usually X and Y will be sets of numbers.

A function is often denoted by the symbol f. The element that the function assigns to the element x is denoted $f(x)$ (read "f of x"). In practice, though, almost everyone speaks interchangeably of the function f or the function $f(x)$.

If $f(x) = y$, x is called the **input** or **argument** and y is called the **output** or **value** of the function at x. Also, x is called the **independent variable** and y the **dependent variable**.

A function may be given by a formula, as in the function $A = x^2$. Because A depends on x, we say that "A is a function of x." Because A depends on only one number, x, it is called a function of a single variable. The first thirteen chapters concern functions of a single variable. The area A of a rectangle depends on its length l and width w; it is a function of *two* variables, $A = lw$. The last five chapters extend calculus to functions of more than one variable and to other functions.

Ways to write and talk about a function

The function that assigns to each argument x the value x^2 is usually described in a shorthand. For instance, we may write $x \mapsto x^2$ (and say "x goes to x^2" or "x is mapped to x^2"). Or we may say simply, "the formula x^2", "the function x^2", or, sometimes, just "x^2." Using this abbreviation, we might say, "How does x^2 behave when x is large?" Some people object to the shorthand

"x^2" because they fear that it might be misinterpreted as the number x^2, with no sense of a general assignment. In practice, the context will make it clear whether x^2 refers to a number or to a function.

EXAMPLE 1 Consider a circle of radius a, as shown in Figure 1.1.3. Let $f(x)$ be the length of chord AB of this circle at a distance x from the center of the circle. Find a formula for $f(x)$.

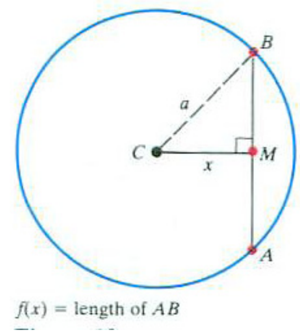


Figure 1.1.3:

SOLUTION We are trying to find how the length $\overline{AB}$ varies as x varies. That is, we are looking for a *formula* for $\overline{AB}$, the length of AB, in terms of x.

Before searching for the formula, it is a good idea to calculate $f(x)$ for some easy inputs. These values can serve as a check on the formula we work out.

In this case $f(0)$ and $f(a)$ can be read at a glance at Figure 1.1.3: $f(0) = 2a$ and $f(a) = 0$. (Why?) Now let us find $f(x)$ for all x in $[0, a]$.

Let M be the midpoint of the chord AB and let C be the center of the circle. Observe that $\overline{CM} = x$ and $\overline{CB} = a$. By the Pythagorean theorem, $\overline{BM} = \sqrt{a^2 - x^2}$. Hence $\overline{AB} = 2\sqrt{a^2 - x^2}$. Thus

$$f(x) = 2\sqrt{a^2 - x^2}.$$

Does the formula give the correct values at $x = 0$ and $x = a$? $\diamond$

Domain and Range

The set of permissible inputs and the set of possible outputs of a function are an essential part of the definition of a function. These sets have special names, which we now introduce.

> **DEFINITION** *(Domain and range)* Let X and Y be sets and let f be a function from X to Y. The set X is called the **domain** of the function. The set of all outputs of the function is called the **range** of the function.

The range is not necessarily all of Y.

When the function is given by a formula, the domain is usually understood to consist of all the numbers for which the formula is defined.

In Example 1 the domain is the closed interval $[0, a]$ and the range is the closed interval $[0, 2a]$. (For interval notation see Appendix A.)

When using a calculator you must pay attention to the domain corresponding to a function key or command. If you enter a negative number as x and press the $\sqrt{x}$-key to calculate the square root of x your calculator will not be happy. It might display an E for "error" or start flashing, the calculator's standard signal for distress. Your error was entering a number not in the domain of the square root function.

Try it. What does your calculator do? Some advanced calculators go into "complex number" mode to handle square roots of negative numbers.

Try it. *No* calculator, however advanced, can permit division by zero.

You can also get into trouble if you enter 0 and press the $1/x$-key. The domain of $1/x$, the reciprocal function, consists of all numbers — except 0.

Graph of a Function

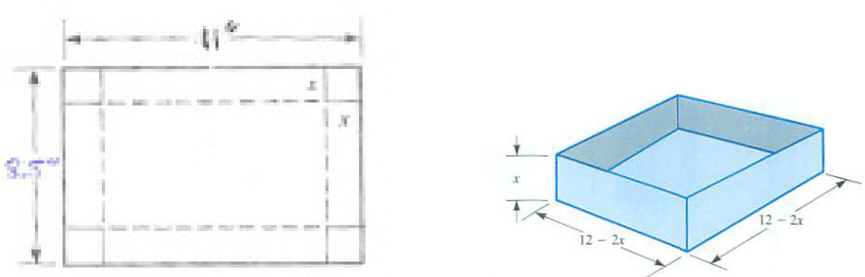
In case both the inputs and outputs of a function are numbers, we can draw a picture of the function, called its **graph**.

> **DEFINITION** *(Graph of a function)* Let f be a function whose inputs and output are numbers. The **graph** of f consists of those points (x, y) in the xy-plane such that $y = f(x)$.

The next example illustrates the usefulness of a graph. We will encounter this function again in Chapter 4.

EXAMPLE 2 A tray is to be made from a rectangular piece of paper by cutting congruent squares from each corner and folding up the flaps. The dimensions of the rectangle are $8\frac{1}{2}'' \times 11''$. Find how the volume of the tray depends on the size of the cutout squares.

SOLUTION Let the side of the cutout square be x inches, as shown in Figure 1.1.4(a). The resulting tray is shown in Figure 1.1.4(b).



(a) (b)

Figure 1.1.4: (a) A rectangular sheet with a square cutout from each corner. (b) The tray formed when the sides are folded up.

The volume $V(x)$ of the tray is the height, x, times the area of the base $(11 - 2x)(8.5 - 2x)$,

$$V(x) = x(11 - 2x)(8.5 - 2x). \tag{1.1.1}$$

The domain of V contains all values of x that lead to an actual tray. This means that x cannot be negative, and x cannot be more than half of the shortest side.

Thus, the largest corners that can be cut out have sides of length 4.25". So, for this tray problem, the domain of interest is only the interval $[0, 4.25]$. Note the peculiar trays that are obtained when $x = 0$ or $x = 4.25$. What are their volumes?

Of course we are free to graph (1.1.1) viewed simply as a polynomial whose domain is $(-\infty, \infty)$.

A short table of inputs and corresponding outputs will help us to sketch the graph. Figure 1.1.5 displays the graph of $V(x)$.

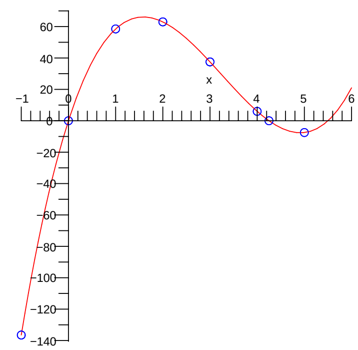


Figure 1.1.5:

x (in)	-1	0	1	2	3	4	4.25	5	6
$V(x)$ (in^3)	-136.5	0	5.85	63	37.5	6	0	-7.5	21

Table 1.1.1:

When $11 - 2x = 0$, that is, when $x = \frac{11}{2} = 5.5$, $V(x) = 0$. When x is greater than $\frac{11}{2}$ all three factors in the formula for $V(x)$ are positive, and $V(x)$ becomes very large for large values of x.

For negative x, two factors in (1.1.1) are positive and one is negative. Thus $V(x)$ is negative and has large absolute value for negative inputs of large absolute value.

Which factor is negative?

Only the part of the graph above the interval $[0, 4.25]$ is meaningful in the tray problem. All other values of x have nothing to do with trays. ◇

If you want to test whether some curve drawn in the xy-plane is the graph of a function, check that each vertical line meets the curve no more than once. If the vertical line $x = a$ meets the curve twice, say at (a, b) and (a, c), there would be the two outputs b and c for the single input a.

Vertical Line Test

The input a is in the domain of f if and only if the vertical line $x = a$ intersects the graph of $y = f(x)$ exactly once. Otherwise, a is not in the domain of f.

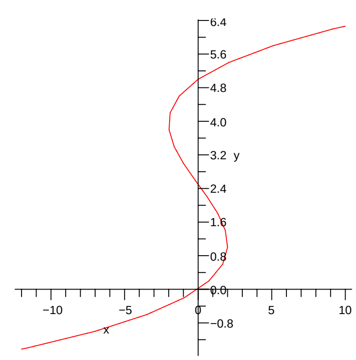


Figure 1.1.6:

For example, Figure 1.1.6 shows a graph that does not pass the vertical line test. The input-output table corresponding to this graph would have three entries for each input x between -2 and 2, two entries for $x = -2$ and $x = 2$ and exactly one entry for each input $x < -2$ or $x > 2$.

In Example 2 the function is described by a single formula, $V(x) = x(11 - 2x)(8.5 - 2x)$. But a function may be described by different formulas for different intervals or individual points in its domain, as in the next example.

EXAMPLE 3 A hollow sphere of radius a has mass M, distributed uniformly throughout its surface. Describe the gravitational force it exerts on a particle of mass m at a distance r from the center of the sphere.

SOLUTION Let $f(r)$ be the force at a distance r from the center of the sphere. In an introductory physics course it is shown that the sphere exerts no force at all on objects in the interior of the sphere. Thus for $0 \leq r < a$, $f(r) = 0$.

The sphere attracts an external particle as though all the mass of the sphere were at its center. Thus, for $r > a$, $f(r) = G\frac{Mm}{r^2}$, where G is the gravitational constant, which depends on the units used for measuring length, time, mass, and force.

It can be shown by calculus that for a particle on the surface, that is, for $r = a$ the force is $G\frac{Mm}{2a^2}$. The graph of f is shown in Figure 1.1.7. ◇

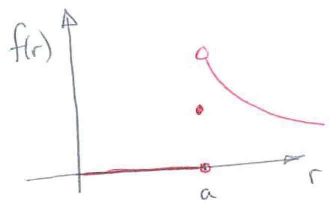


Figure 1.1.7:

The formula describing the function in Example 3 changes for different parts of its domain.

$$f(r) = \begin{cases} 0 & \text{if } 0 \leq r < a \\ \frac{GMm}{2a^2} & \text{if } r = a \\ \frac{GMm}{r^2} & \text{if } r > a \end{cases}$$

Such a function is called a **piecewise-defined function**.

In a graph that consists of several different pieces, such as Figure 1.1.7, the presence of a point on the graph of a function is indicated by a solid dot (•) and the absence of a point by a hollow dot (◦).

Inverse Functions

If you know a particular output of the function $f(x) = x^3$ you can figure out what the input must have been. For instance, if $x^3 = 8$, then $x = 2$ – you can go backwards from output to input. However, you cannot do this with the function $f(x) = x^2$. If you are told that $x^2 = 25$, you do not know what x is. It can be 5 or -5. However, if you are told that $x^2 = 25$ and that x is positive, then you know that x is 5.

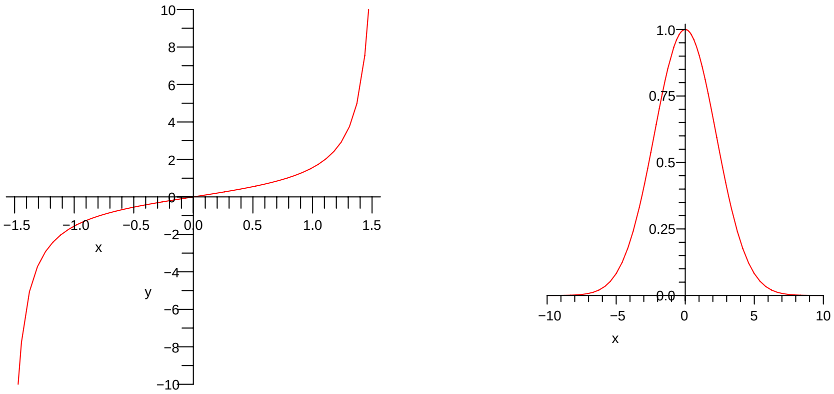
This brings us to the notion of a one-to-one function.

DEFINITION *(One-to-One Function)* A function f that does not assign the same output to two different inputs is **one-to-one**. That is, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

DEFINITION *(Inverse Function)* If f is a one-to-one function, the **inverse function** is the function g that assigns to each output of f the corresponding input. That is, if $f(x) = y$ then $g(y) = x$.

Horizontal Line Test

The graph of a one-to-one function never meets a horizontal line more than once. (See Figure 1.1.8.)



(a) (b)

Figure 1.1.8: The function in (a) is one-to-one as it passes the horizontal line test. The function in (b) does not pass the horizontal line test, so is not one-to-one.

The function $f(x) = x^3$ is one-to-one on the entire real line. A few entries in the tables for $f(x)$ and its inverse function are shown in Table 1.1.2(a) and (b), respectively.

input	1	2	$\frac{1}{2}$	3	-2
output	1	8	$\frac{1}{8}$	27	-8

(a)

input	1	8	$\frac{1}{8}$	27	-8
output	1	2	$\frac{1}{2}$	3	-2

(b)

Table 1.1.2: (a) Table of input and output value for $f(x) = x^3$. (b) Table of input and output values for the inverse of $f(x) = x^3$.

In this case an explicit formula for the inverse function can be found algebraically: if $y = x^3$ then $y^{1/3} = (x^3)^{1/3} = x$. Then $x = y^{1/3}$. Since it is customary to use the x-axis for the input and the y-axis for the output, it is convenient to rewrite $x = y^{1/3}$ as $y = x^{1/3}$. (Both say the same thing: "The output is the cube root of the input.".)

By the way, an inverse of a one-to-one function may not be given by a nice formula. For instance, $f(x) = 2x + \cos(x)$ is one-to-one, as will be easily shown in Chapter 4. However, the inverse function is not described by a convenient formula. Happily, we do not need to deal with an explicit formula for this particular inverse function.

The Graph of an Inverse Function

Suppose you know the graph of a one-to-one function. Then there is an easy way to draw the graph of the inverse function.

If (a, b) is a point on the graph of the function f, that is, $b = f(a)$, then (b, a) is a point on the graph of inv$f$, shown in Figure 1.1.9.

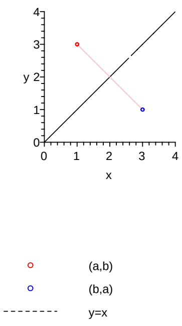


Figure 1.1.9: The point (b, a) is obtained by reflecting (a, b) around the line $y = x$.

Notation: The use of inv$f$ to denote the inverse function of f is based on the fact that many calculators have a button marked inv to indicate the inverse of a function. The mathematical notation for the inverse function of f is f^{-1} or inv$f$. *Note that the -1 is not an exponent, and in general the inverse and reciprocal functions are different: f^{-1} is not equal to $\frac{1}{f}$.*

EXAMPLE 4 Draw the graphs of (a) the inverse of the cubing function given by $f(x) = x^3$, and (b) the squaring function $g(x) = x^2$ restricted to $x \geq 0$.

SOLUTION See Figure 1.1.10. ◇

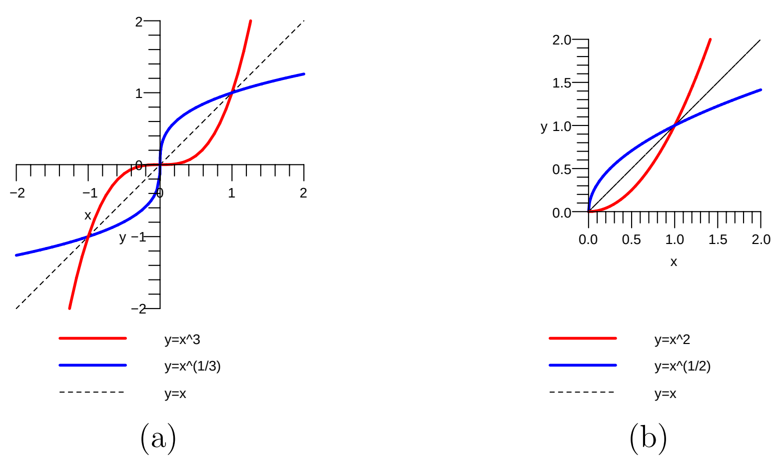


Figure 1.1.10: (a) Plots of $f(x) = x^3$ and $f^{-1}(x) = x^{1/3}$. (b) Plots of $g(x) = x^2$ $(x \geq 0)$ and $g^{-1}(x) = \sqrt{x}$.

EXAMPLE 5 Let $m \neq 0$ and b be constants and $f(x) = mx + b$. Show that f is one-to-one and describe its inverse function.

SOLUTION If $f(x_1) = f(x_2)$ we have

$$\begin{aligned} mx_1 + b &= mx_2 + b & \\ mx_1 &= mx_2 & \text{subtract } b \text{ from both sides} \\ x_1 &= x_2 & \text{divide both sides by } m \neq 0 \end{aligned}$$

Because $f(x_1) = f(x_2)$ only when $x_1 = x_2$, f is one-to-one.

This problem can also be analyzed graphically. The graph of $y = f(x)$ is the line with slope m and y-intercept b. (See Figure 1.1.11.) It passes the horizontal line test.

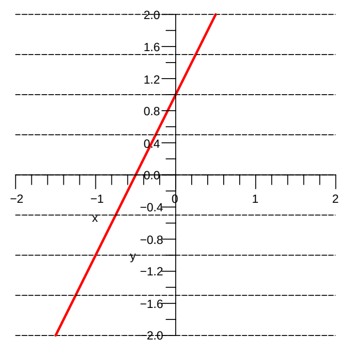


Figure 1.1.11:

To find the inverse function, solve the equation $y = f(x)$ to express x in terms of y:

$$\begin{aligned} y &= mx + b & \\ y - b &= mx & \text{subtract } b \text{ from both sides} \\ \tfrac{y-b}{m} &= x & \text{divide by } m \neq 0 \\ x &= \tfrac{y}{m} - \tfrac{b}{m} & \text{move } x \text{ to left-hand side} \\ y &= \tfrac{x}{m} - \tfrac{b}{m} & \text{interchange } x \text{ and } y. \end{aligned}$$

Reversing the roles of x and y in the final step is done only to present the inverse function in a form where the input is called x and the output is called y. Thus the inverse function has the formula

$$f^{-1}(x) = \frac{x}{m} - \frac{b}{m}.$$

The graph of the inverse function is also a line; its slope is $1/m$, the reciprocal of the slope of the original line, and its y-intercept is $-b/m$. (See Figure 1.1.12.) ◇

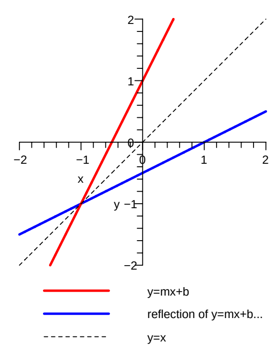


Figure 1.1.12:

Decreasing and Increasing Functions

There is another way to check whether a function is one-to-one on an interval. It uses the following concepts.

A function is **increasing** on an interval if whenever x_1 and x_2 are in the interval and x_2 is greater than x_1, then $f(x_2)$ is greater than $f(x_1)$. As you move along the graph of f from left to right, you go up. This is shown in Figure 1.1.14(a).

In the case of a **decreasing** function, you go down as you move from left to right: if $x_2 > x_1$ then $f(x_2) < f(x_1)$. (See Figure 1.1.14(b).)

For instance, consider $f(x) = \sin(x)$, whose graph is shown in Figure 1.1.13. On the interval $[-\pi/2, \pi/2]$ the values of $\sin(x)$ increase. On the interval

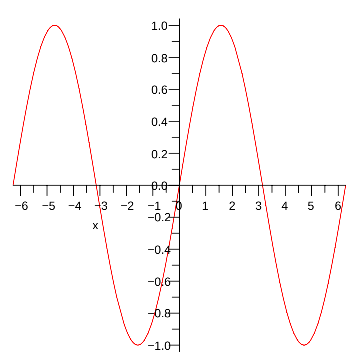
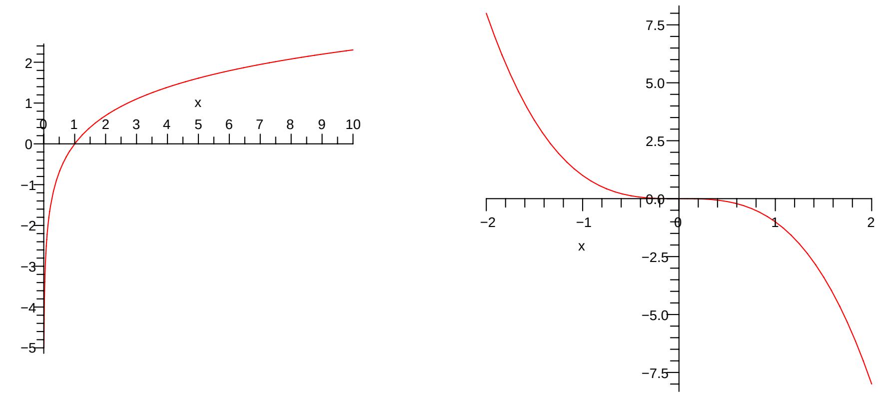


Figure 1.1.13:



(a) (b)

Figure 1.1.14: Graph of (a) an increasing function and (b) a decreasing function.

$[\pi/2, 3\pi/2]$ the values of $\sin(x)$ decrease. The function x^3 increases on its entire domain $(-\infty, \infty)$.

A **monotonic** function is a function that is either only increasing or only decreasing. A monotone function always passes the Horizontal Line Test, as the next example illustrates.

Monotone is from the Greek, *mono*=single, *tonos*=tone, which also gives us the word 'monotonous').

EXAMPLE 6 For $k \neq 0$ and $x > 0$, x^k is a monotonic function. The inverse of x^k is $x^{1/k}$. If $k = 0$, we have a constant function, $x^0 = 1$. The constant function does not pass the Horizontal Line Test; therefore it has no inverse. $\diamond$

Because strict inequalities are used in the definitions of increasing and decreasing, we sometimes say these functions are *strictly increasing* or *strictly decreasing* on an interval. A function f is said to be **non-decreasing** on an interval if whenever x_1 and x_2 are in the interval and x_2 is greater than x_1, then $f(x_2) \geq f(x_1)$. The graph of a non-decreasing function is increasing except on intervals where it is constant. Likewise, f is **non-increasing** on an interval if whenever x_1 and x_2 are in the interval and x_2 is greater than x_1, then $f(x_2) \leq f(x_1)$.

The sign of a function's outputs provides another way to describe a function. A function that has only positive outputs is called a **positive function**; for instance, 2^x. A **negative function** has only negative outputs; for instance, $\frac{-1}{1+x^2}$. A **non-negative function** has outputs that are either positive or zero; for instance x^2. The outputs of a **non-positive function** are either negative or zero, for instance, $\sin(x) - 1$.

Summary

This section introduced concepts that will be used throughout the coming chapters: function, domain, range, graph, piecewise-defined function, one-to-one functions, inverse functions, increasing functions, decreasing functions, monotonic functions, non-decreasing functions, non-increasing functions, positive functions, negative functions, non-negative functions, and non-positive functions.

Two important observations are that every monotone function has an inverse function and the graph of the inverse function is the reflection across the line $y = x$ of the graph of the original function.

A function can be described in several ways: by a formula, such as $V(x) = x(11 - 2x)(8.5 - 2x)$, by a table of values, or by words, such as "the volume of a tray depends on the size of the cut-out square."

EXERCISES for Section 1.1 *Key:* R–routine, M–moderate, C–challenging

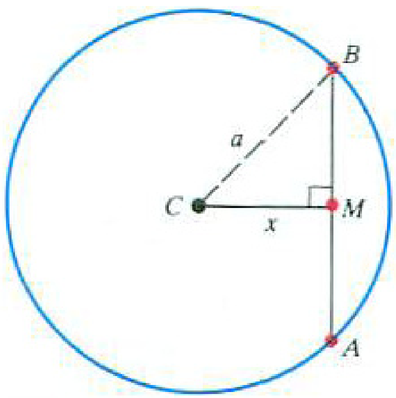


Figure 1.1.15: Exercises 1 to 4. ARTIST: Please add θ to denote the angle BCM.

Exercises 1 to 4 refer to Figure 1.1.15.

1.[R] Express the area of triangle ABC as a function of $x = \overline{CM}$

2.[R] Express the perimeter of triangle ABC as a function of x.

3.[R] Express the area of triangle ABC as a function of θ.

4.[R] Express the perimeter of triangle ABC as a function of θ.

In Example 2 a tray was formed from an $8\frac{1}{2}$" by 11" rectangle by removing squares from the corners. Find and graph the corresponding volume function for trays formed from sheets with the dimensions given in Exercises 5 to 8.

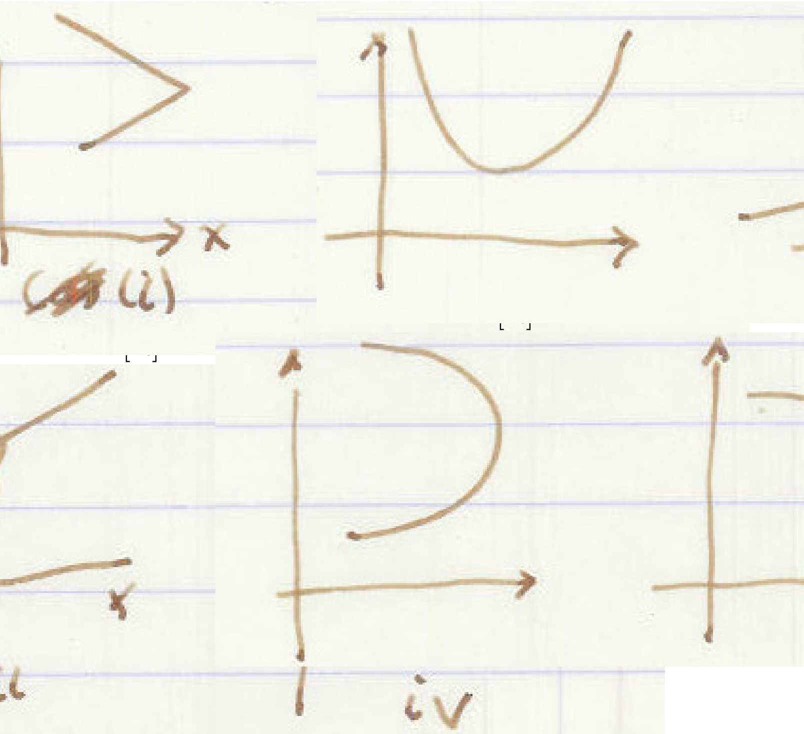
5.[R] 4" by 13"

6.[R] 5" by 7"

7.[R] 6" by 6"

8.[R] 5" by 5"

In Exercises 9 and 10 decide which curves are graphs of (a) functions, (b) increasing functions, and (c) one-to-one functions.



11.[R] Let $f(x) = x^3$.

(a) Fill in this table

x	0	1/4	1/2	$-1/4$	$-1/2$	1	2
x^3							

(b) Graph f.

(c) Use the table in (a) to find seven points on the graph of f^{-1}.

(d) Graph f^{-1} (use the same axes as in (b)).

12.[R] Let $f(x) = \cos(x)$, $0 \leq x \leq \pi$ (angles in radians).

(a) Fill in this table

x	0	$\pi/6$	$\pi/4$	$2\pi/3$	$\pi/2$	$3\pi/4$	π
$\cos(x)$							

(b) Graph f.

(c) Use the table in (a) to find seven points on the graph of $\mathsf{inv}\cos$.

(d) Graph $\mathsf{inv}\cos$ (use the same axes as in (b)).

In Exercises 13 to 18 the functions are one-one. Find the formula for each inverse function, expressed in the form $y = g(x)$, so that the independent variable is labeled x. NOTE: If you have trouble with the use of logarithms in Exercise 17 or Exercise 18, read Appendix D.

13.[R] $y = 3x - 2$

14.[R] $y = x/2 + 7$

15.[R] $y = x^5$

16.[R] $y = 3\sqrt{x}$

17.[R] $y = 3^x$

18.[R] $y = 5(2^x)$

In Exercises 19 to 23 the slope of line L is given. Let L' be the reflection of L across the line $y = x$. What is the slope of the reflected line, L'? In each case sketch a possible L and its reflection, L'.

19.[R] L has slope 2.

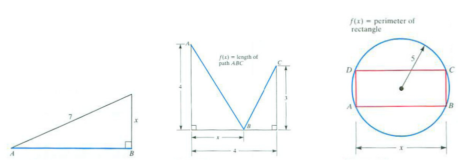
20.[R] L has slope 1.

21.[R] L has slope 1/10.

22.[R] L has slope $-1/3$.

23.[R] L has slope -2.

In Exercises 24 to 33 state the formula for the function f and give the domain of the function.



(a) (b) (c)

Figure 1.1.16:

24.[R] $f(x)$ is the perimeter of a circle of radius x.

25.[R] $f(x)$ is the area of a disk of radius x.

26.[R] $f(x)$ is the perimeter of a square of side x.

27.[R] $f(x)$ is the volume of a cube of side x.

28.[R] $f(x)$ is the total surface area of a cube of side x.

29.[R] $f(x)$ is the length of the hypotenuse of the right triangle whose legs have lengths 3 and x.

30.[M] $f(x)$ is the length of the side AB in the triangle in Figure 1.1.16(a).

31.[M] For $0 \le x \le 4$, $f(x)$ is the length of the path from A to B to C in Figure 1.1.16(b).

32.[M] For $0 \le x \le 10$, $f(x)$ is the perimeter of the rectangle $ABCD$, one side of which has length x, inscribed in the circle of radius 5 shown in Figure 1.1.16(c).

33.[C] A person at point A, two miles from shore in a lake, is going to swim to the shore ST and then walk to point B, five miles from the shore. She swims at 1.5 miles per hour and walks at 4 miles per hour. If she reaches the shore at point P, x miles from S, let $f(x)$ denote the time for her combined swim and walk. Obtain a formula for $f(x)$. (See Figure 1.1.19(a).)

34.[M] A camper at A will walk to the river, put some water in a pail at P, and take it to the campsite at B.

(a) Express the distance $\overline{AP} + \overline{PB}$ as a function of x.

(b) Where should P be located to minimize the length of the walk, $\overline{AP} + \overline{PB}$? (See Figure 1.1.17.) Hint: Reflect B across the line L.

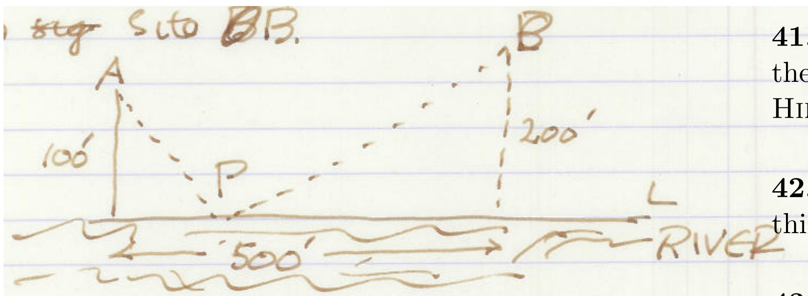


Figure 1.1.17: Sketch of situation in Exercise 34.
Note: A geometric trick solved (b). Chapter 4 develops a general procedure for finding the maximum or minimum of a function.

In Exercises 35 to 39 give (a) three functions that satisfy the equation for all positive x and y and (b) one function that does not.

35.[M] $f(x+y) = f(x) + f(y)$

36.[M] $f(x+y) = f(x)f(y)$

37.[M] $f(xy) = f(x) + f(y)$

38.[M] $f(xy) = f(x)f(y)$

39.[M] $f(x) = f(y)$

40.[M] The cost of life insurance depends on whether the person is a smoker or a non-smoker. The following chart lists the annual cost for a male for a million-dollar life insurance policy.

age (yrs)	20	30	40	50	60	70	80
cost for smoker ($)	1150	1164	1944	4344	9864	26500	104600
cost for non-smoker ($)	396	396	600	1490	3684	10900	41600

Note: A "smoker" is a person who has used tobacco during the previous three years.

(a) Plot the data and sketch the graphs on the same axes for both groups of males.

(b) A smoker at age 20 pays as much as a non-smoker of about what age?

(c) A smoker pays about how many times as much as a non-smoker of the same age?

41.[M] Find the function that gives the radius of the largest circle that fits in a one by x rectangle. Hint: This will be a piecewise-defined function.

42.[M] If f is an increasing function, what, if anything, can be said about f^{-1}?

43.[M] On a typical summer day in the Sacramento Valley the temperature is at a minimum of 60° at 7a.m. and a maximum of 95° at 4p.m..

(a) Sketch a graph that shows how the temperature may vary during the twenty-four hours from midnight to midnight.

(b) A closed shed with little insulation is in the middle of a treeless field. Sketch a graph that shows how the temperature inside the shed may vary during the same period.

(c) Sketch a graph that shows how the temperature in a well-insulated house may vary. Assume that in the evening all the windows and skylights are opened when the outdoor temperature equals the indoor temperature, and closed in the morning when the two temperatures are again equal.

Note: Use the same set of axes for all three graphs.

44.[M] The monthly average air and water temperatures in Myrtle Beach, SC, are shown in Table 1.1.3.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	A
Air Temp (°)	56	60	68	76	83	88	91	
Water Temp (°)	51	52	57	62	69	77	81	

Table 1.1.3: Source: `http://www.myrtle-beach-resort.com/weather.htm`

NOTE: Assume, for convenience, that the temperatu in the table are the temperatures on the first day each month.

(a) Sketch a graph that shows how the water te perature may vary during one calendar year, th is, from January 1 through December 31.

(b) Sketch a graph that shows how the difference between the air and water temperatures may vary during one calendar year. During what month is the temperature difference greatest? least?

(c) During February, the water temperature increases 5° in 28 days so the average daily change is $5/28 \approx 0.1786°$/day. For each month, estimate the average daily change in the water temperature from one day to the next. During which month is this daily change greatest? least?

(d) Repeat (b) and (c) for the air temperature data.

45.[M] This problem grew out of a question raised by the daughter of one of the authors, Rebecca Stein-Wexler, when cutting cloth for a dress. She wanted to cut out two congruent semicircles from a long strip of fabric 44 inches wide, as shown in Figure 1.1.18. The radius of the semicircles determines d, the length of fabric used, $d = f(r)$.

(a) Draw a picture to show that $f(22) = 44$.

(b) For $0 \leq r \leq 22$, determine d as a function of r, $d = f(r)$.

(c) For $22 \leq r \leq 44$, determine d as a function of r, $d = f(r)$.

(d) Obtain an equation expressing r as a function of d.

(e) She had 104 inches of fabric, and guesed that the largest semicircle she could cut set has a radius of about 30 inches. Use (c) to see how good her guess is.

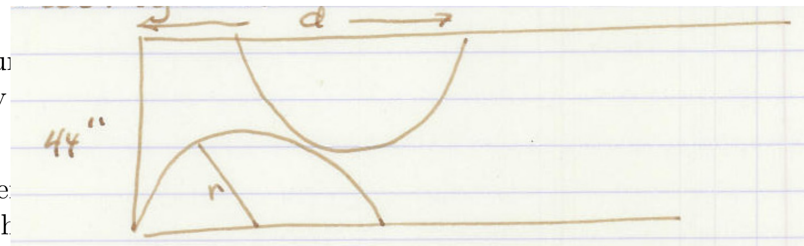


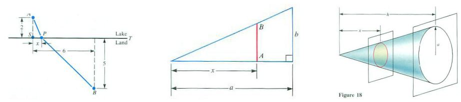
Figure 1.1.18: Exercise 45.

46.[C] Let $f(x)$ be the length of the segment AB in Figure 1.1.19(b).

(a) What are $f(0)$ and $f(a)$?

(b) What is $f(a/2)$?

(c) Find the formula for $f(x)$ and explain your solution.



(a) (b) (c)

Figure 1.1.19:

47.[C] Let $f(x)$ be the area of the cross-section of aright circular cone shown in Figure 1.1.19(c).

(a) What are $f(0)$ and $f(h)$?

(b) Find a formula for $f(x)$ and explain your solution.

48.[C] The cost of a ride in a New York city taxi is described by this formula: At the start the meter reads \$2.50. For every fifth of a mile, 40 cents is added. Graph the cost as a function of distance travelled. NOTE: The cost also depends on other factors. For every two minutes stopped in traffic, 40 cents is added. During the evening rush, 4–8 pm, there is a surcharge of one dollar. Between 8 pm and 6 am there is a surcharge of 50 cents. So the cost, which depends on distance travelled, time stopped, and time of day, is actually a function of three variables.)

49.[C] Let $g(d)$ be the radius of the largest pair of semicircles with diameters on the edge of the fabric, if the fabric is d inches long and 44 inches wide. The domain of g is $(0, \infty)$. Find g and graph it. NOTE: This is related to Exercise 45.

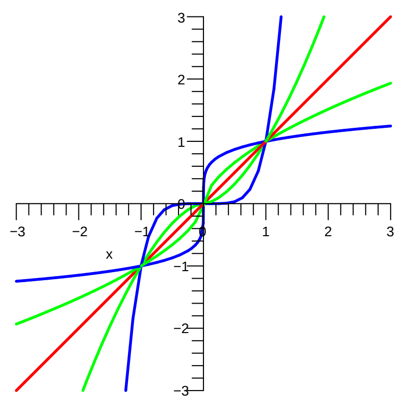
1.2 The Basic Functions of Calculus

This section describes the basic functions in calculus. In the next section you will see how to use them as building blocks to build more complicated functions.

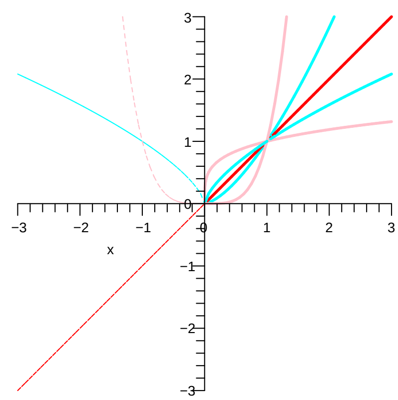
The Power Functions

The first group of functions consists of the **power functions** x^k where the exponent k is a fixed non-zero number and the base x is the input. If k is an odd integer, then x^k has an inverse, $x^{1/k}$, another power function. If k is an even integer and we restrict the domain of x^k to the positive numbers, then it is one-to-one, and has an inverse, again $x^{1/k}$, with, again, a domain consisting of all positive numbers.

In Section 1.1 it was shown that the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ for all x. Notice, however, $g(x) = x^4$ does not pass the horizontal line test unless the domain is restricted to, say, nonnegative inputs ($x \geq 0$). Thus, the inverse of $g(x) = x^4$ is $g^{-1}(x) = x^{1/4}$ only for $x \geq 0$.



(a)



(b)

Figure 1.2.1: Graphs of power functions. (a) x^k for $k = 1$ (red), 5 (blue), 1/5 (blue), 5/3 (green), and 3/5 (green). (b) x^k for $k = 1$ (red), 4 (pink), 1/4 (pink), 3/2 (aqua), and 2/3 (aqua). Note that the pairs of blue and green graphs are inverses in (a), as are the pairs of (solid) pink and aqua graphs in (b). In (b) the graphs of x^4 and $x^{2/3}$ pass the horizontal line test only for $x \geq 0$, and the graphs of $x^{1/4}$ and $x^{3/2}$ are defined only for $x \geq 0$.

OBSERVATION *(Inverses of Power Functions)*

1. The inverse of a power function is another power function.

2. When $k = 0$, we obtain the function x^0, which is constant (with all outputs equal to 1), the very opposite of being one-to-one. Constant functions are discussed in more detail in Section 1.3.
3. When the exponent k is an even integer or a rational number (in lowest terms) whose numerator is even (2/3, 4/7, etc.) the graph of $y = x^k$ will not pass the horizontal line test unless the domain is reduced, typically by restricting it to $[0, \infty)$.

The Exponential and Logarithm Functions

Next we have the **exponential functions** b^x where the base b is fixed and the exponent x is the input. The inverses of exponential functions are not exponential functions. The inverses are called **logarithms** and are the next class of functions that we will consider. (If you need a review of logarithms and their properties, please see Section 1.5.)

Consider a function of the form b^x, where b is positive and fixed. In order to be concrete, let's take the case $b = 2$, that is, $f(x) = 2^x$.

As x increases, so does 2^x. So the function 2^x has an inverse function. In other words, if $y = 2^x$, then if we know the output y we can determine the input x, the exponent, uniquely. For instance, if $2^x = 8$ then $x = 3$. This is expressed as $3 = \log_2 8$ and it read as "the logarithm of 8, base 2, is 3." If $y = b^x$, then we write $x = \log_b y$.

Since we usually denote the independent variable (the input or argument) by x, and the dependent variable (the output, or value) by y, we will rewrite this as $y = \log_b(x)$.

The table of easy values of $\log_2(x)$ in Table 1.2.1 will help us graph $y = \log_2(x)$. Putting a smooth curve through the seven points in Table 1.2.1 yields the graph in Figure 1.2.2.

x	1	2	4	8	1/2	1/4	1/8
$\log_2(x)$	0	1	2	3	-1	-2	-3

Table 1.2.1: Table of easy values of $y = \log_2(x)$.

As x increases, $\log_2(x)$ grows very slowly. For instance $\log_2 1024 = 10$, as every computer scientist knows. For x between 0 and 1, $\log_2(x)$ is negative. As x moves from 1 towards 0, $|\log_2(x)|$ grows very large. For instance, $\log_2 \frac{1}{1024} = -10$.

Because $\log_2(x)$ is the inverse of the function 2^x, we could have sketched the graph of $y = \log_2(x)$ by first sketching the graph of $y = 2^x$ and reflecting it around the line $y = x$.

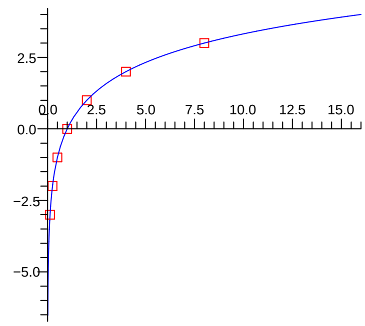


Figure 1.2.2: Plot of $y = \log_2(x)$ based on data in Table 1.2.1.

For any positive base b, $\log_b(x)$ is defined similarly. For x and b both positive numbers, the logarithm of x to the base b, denoted $\log_b(x)$, is the power to which we must raise b to obtain x. By the very definition of the logarithm

$$b^{\log_b(x)} = x.$$

(Whenever you see "$\log_b(x)$" you should think, "Ah, ha! The fancy name for an exponent.")

The Trigonometric Functions and Their Inverses

So far we have the power functions, x^k, the exponential functions, b^x, and the logarithm functions, $\log_b(x)$. The last major group of important functions consists of the trigonometric functions, $\sin(x)$, $\cos(x)$, $\tan(x)$, and their inverses (after we shrink their domains to make the functions one-to-one).

$\sin(x)$ and its inverse

In calculus we generally measure angles in radians. See also Appendix E.

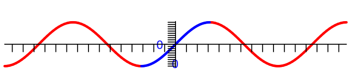


Figure 1.2.3:

The graph of the **sine function** $\sin(x)$ has period 2π and is shown in Figure 1.2.3. The range is $[-1, 1]$. On the domain $[-\pi/2, \pi/2]$, $\sin(x)$ is increasing and its values for these inputs already sweep out the full range, $[-1, 1]$.

When we restrict the domain of the function $\sin(x)$ to $[-\pi/2, \pi/2]$ it is a one-to-one function with range $[-1, 1]$. This means the sine function has an inverse with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$. The inverse sine function is denoted by $\arcsin(x)$, $\sin^{-1}(x)$, or $\mathsf{inv}\sin(x)$.

Let's stop for a moment to summarize our findings: For x in $[-1, 1]$, $\arcsin(x)$ is the angle in $[-\pi/2, \pi/2]$ whose sine is x. In equations:

$$y = \arcsin(x) \iff \sin(y) = x.$$

For instance, $\arcsin(1) = \pi/2$ because the angle in $[-\pi/2, \pi/2]$ whose sine is 1 is $\pi/2$. Similarly, $\sin^{-1}(1/2) = \pi/6$, $\mathsf{inv}\sin(0) = 0$, $\arcsin(-1/2) = -\pi/6$, $\sin^{-1}(-1) = -\pi/2$. Drawing a unit circle will display these facts, as Figure 1.2.4 illustrates.

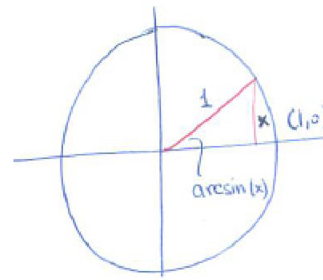
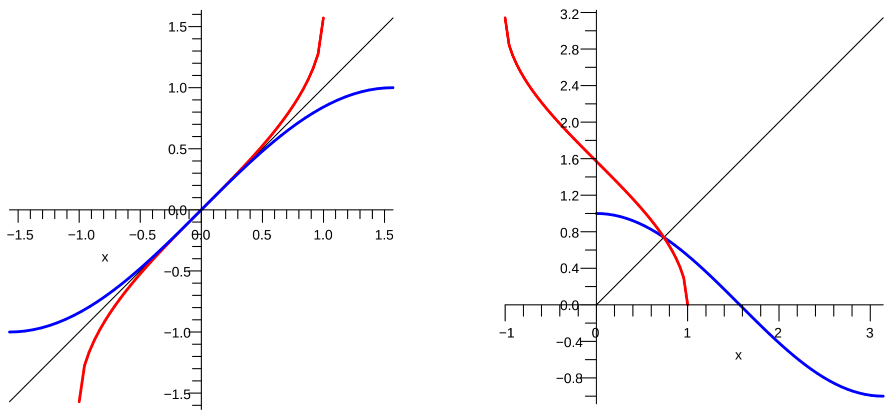


Figure 1.2.4:

We could graph $y = \arcsin(x)$ with the aid of these five values. However, it's easier just to reflect the graph of $y = \sin(x)$ around the line $y = x$. (See Figure 1.2.5(a).)



(a) (b)

Figure 1.2.5: (a) The graph of $y = \arcsin(x)$ (red) is the graph of $y = \sin(x)$ (blue), with domain restricted to $[-\pi/2, \pi/2]$, reflected around the line $y = x$. (b) The graph of $y = \arccos(x)$ (red) is the graph of $y = \cos(x)$ (blue), with domain restricted to $[0, \pi]$, around the line $y = x$.

$\cos(x)$ and its inverse

The graph of the **cosine function** $\cos(x)$ is shown in Figure 1.2.6.

It is clearly not one-to-one, even if we restrict the domain to the domain used for $\sin(x)$, namely $[-\pi/2, \pi/2]$. In this case note that $\cos(x)$ is a decreasing function on $[0, \pi]$. So the cosine function is one-to-one on $[0, \pi]$. Moreover, the values of $\cos(x)$ for x in $[0, \pi]$ sweep out all possible values of the cosine function, namely $[-1, 1]$.

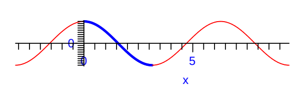


Figure 1.2.6: ARTIST: Please add a more visible vertical axis.

Because $\cos(x)$ is a one-to-one function on the domain $[0, \pi]$, it has an inverse function, called $\arccos(x)$, $\mathsf{inv}\cos(x)$, or simply $\cos^{-1}(x)$. Each of these is short for "the angle in $[0, \pi]$ whose cosine is x". For instance, $\arccos(0) = \pi/2$, $\cos^{-1}(1) = 0$, and $\mathsf{inv}\cos(-1) = \pi$. Moreover, because the range of the cosine function is the closed interval $[-1, 1]$, the domain of arccos is $[-1, 1]$. Figure 1.2.5(b) shows that the graph of $\arccos(x)$ is obtained by reflecting the graph of $\cos(x)$, with domain $[0, \pi]$, around the line $y = x$.

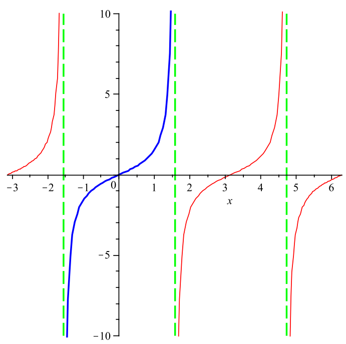


Figure 1.2.7:

$\tan(x)$ and its inverse

The range of the function $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is $(-\infty, \infty)$, as Figure 1.2.7 shows.

When the inputs are restricted to $(-\pi/2, \pi/2)$, $\tan(x)$ is one-to-one, and therefore has an inverse function, denoted $\arctan(x)$, $\tan^{-1}(x)$, or $\mathsf{inv}\tan(x)$. The domain of the inverse tangent function is $(-\infty, \infty)$ and its range is $(-\pi/2, \pi/2)$.

For instance, $\tan^{-1}(0) = 0$, $\mathsf{inv}\tan(1) = \pi/4$, and as x increases, $\arctan(x)$ approaches $\pi/2$. Also, $\arctan(-1) = -\pi/4$, and when x is negative and becomes ever more negative (that is, $|x|$ becomes bigger and bigger) $\arctan(x)$ approaches $-\pi/2$. Figure 1.2.8 is the graph of $\arctan(x)$. It is the reflection of the blue part of the graph in Figure 1.2.7 across the line $y = x$. (See Figure 1.2.8.)

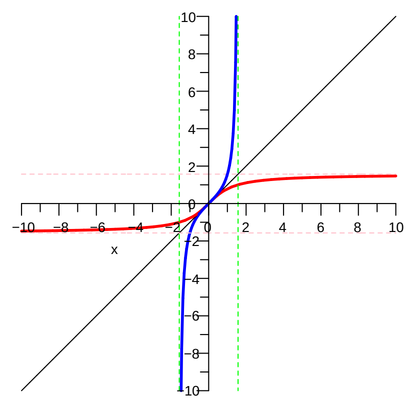


Figure 1.2.8: ARTIST: Please label the two curves as $y = \tan(x)$ and $y = \arctan(x)$.

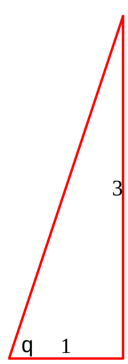


Figure 1.2.9: The traditional symbol for angles is the Greek letter θ (pronounced "theta"). ARTIST: Check that the angle is labeled as θ, not q.

EXAMPLE 1 Evaluate

$$\text{(a) } \sin(\sin^{-1}(0.3)), \quad \text{(b) } \sin(\tan^{-1}(3)), \quad \text{and (c) } \tan(\cos^{-1}(0.4)).$$

SOLUTION

(a) The expression $\sin^{-1}(0.3)$ is short for the angle in the interval $[-\pi/2, \pi/2]$ whose sine is 0.3. So, the sine of $\sin^{-1}(0.3)$ is 0.3.

(b) To find $\sin(\tan^{-1}(3))$, first draw the angle θ whose tangent is 3 (and lies in the interval $[-\pi/2, \pi/2]$. Figure 1.2.9 shows a simple way to draw this angle. To find the sine of θ, recall that sine equals "opposite/hypotenuse." By the Pythagorean Theorem, the hypotenuse is $\sqrt{3^2 + 1^2} = \sqrt{10}$. Thus, $\sin(\tan^{-1} 3) = 3/\sqrt{10}$.

(c) To evaluate $\tan(\cos^{-1}(0.4))$, first draw an angle whose cosine is $0.4 = \frac{2}{5}$, as in Figure 1.2.10, which is based on the fact that cosine equals equals "$\frac{\text{adjacent}}{\text{hypotenuse}}$." To find the tangent of this angle, we need the length of the other leg in Figure 1.2.10. By the Pythagorean Theorem this length is $\sqrt{5^2 - 2^2} = \sqrt{21}$.

From the relation $\tan(\theta) = \text{opposite}/\text{adjacent}$, we conclude that

$$\tan(\cos^{-1}(0.4)) = \sqrt{21}/2 \approx 2.291.$$

◇

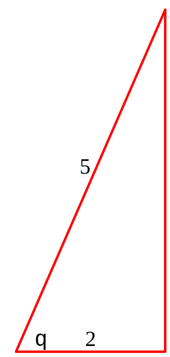


Figure 1.2.10: ARTIST: Check that the angle is labeled as θ, not q.

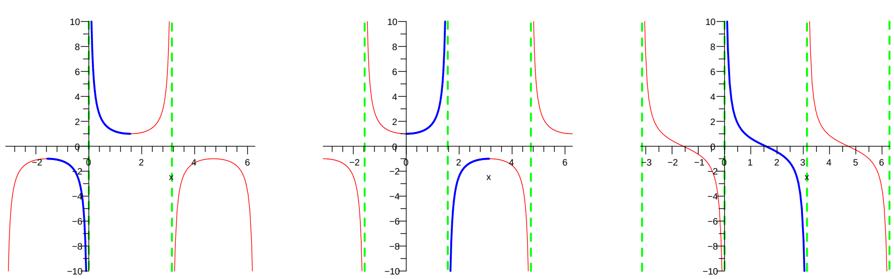
WARNING *(Notation for Inverse Functions)* The notation "$\sin^{-1}(x)$ can be confusing. It may be read as $(\sin(x))^{-1}$, the reciprocal of $\sin(x)$. After all, $\sin^2(x)$ means $(\sin(x))^2$. Though the notation "$\sin^{-1}(x)$" is shorter than "$\arcsin(x)$," we prefer the latter to avoid the risk of misinterpretation. Similar comments apply to $\tan^{-1}(x)$ and $\arctan(x)$ and to $\cos^{-1}(x)$ and $\arccos(x)$.

$\csc(x)$, $\sec(x)$, and $\cot(x)$ and their inverses

The **cosecant**, **secant**, and **cotangent** functions are defined in terms of the sine and cosine functions:

$$\csc(x) = \frac{1}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \text{and} \quad \cot(x) = \frac{\sin(x)}{\cos(x)}.$$

While we could write $\csc(x) = (\sin(x))^{-1}$, we do not because of the possible confusion with $\sin^{-1}(x) = \arcsin(x)$. Each of these functions is defined only when the denominator is not zero. Figure 1.2.11 shows their graphs.



(a) (b) (c)

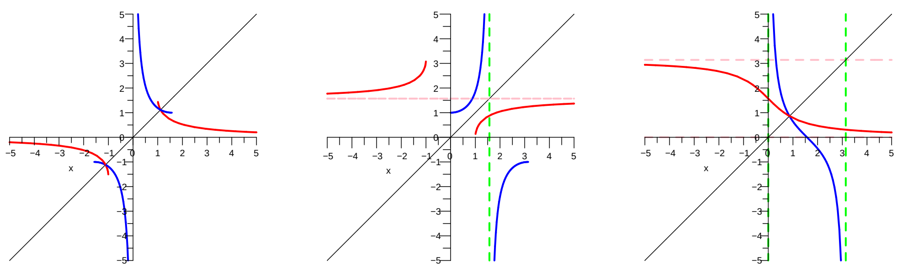
Figure 1.2.11: The graphs of (a) the cosecant, (b) the secant, and (c) the cotangent functions. ARTIST: Please add "cosecant,", "secant," and "cotangent" above each of graph, respectively.

Note that $|\sec(x)| \geq 1$ and $|\csc(x)| \geq 1$. In each case the range consists of two separate intervals: $[1, \infty)$ and $(-\infty, -1]$.

These three functions have inverses, when restricted to appropriate intervals. Table 1.2.2 contains a summary of the three inverse functions, $\csc^{-1} x$, $\sec^{-1} x$, and $\tan^{-1} x$. Figure 1.2.12 shows the graphs of csc, sec, and cot and their inverses.

function	domain (input)	range (output)
$\csc^{-1}(x)$	$(-\infty, -1]$ and $[1, \infty)$	all of $[-\pi/2, \pi/2]$ except 0
$\sec^{-1}(x)$	$(-\infty, -1]$ and $[1, \infty)$	all of $[0, \pi]$ except 0, that is $(0, \pi]$
$\cot^{-1} x$	$(-\infty, \infty)$	the open interval $(0, \pi)$

Table 1.2.2: Summary of the inverse cosecant, inverse secant, and inverse cotangent functions.



(a) (b) (c)

Figure 1.2.12: Graphs of (a) $y = \csc(x)$ and $y = \csc^{-1}(x)$, (b) $y = \sec(x)$ and $y = \sec^{-1}(x)$, and (c) $y = \cot(x)$ and $y = \cot^{-1}(x)$. Notice how the inverse function is the reflection of the original function across the line $y = x$.

Summary

This section reviewed the basic functions in calculus, x^k, b^x, $\sin(x)$, $\cos(x)$, $\tan(x)$, and their inverses. $\log_b(x)$, $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$. (The inverse of x^k, $k \neq 0$, is just another power function $x^{1/k}$).

The functions that may be hardest to have a feel for are the logarithms. Now, $\log_2(x)$ is typical of $\log_b(x)$, $b > 1$. These are its key features:

- its graph crosses the x-axis at $(1, 0)$ because $\log_2(1) = 0$ $(2^0 = 1)$,
- it is defined only for positive inputs, that is, the domain of $\log_2$ is $(0, \infty)$, because only positive numbers can be expressed in the form 2^x,

- it is an increasing function,
- it grows very slowly as the argument increases: $\log_2(8) = 3$, $\log_2(16) = 4$, $\log_2(32) = 5$, $\log_2(64) = 6$, and $\log_2(1024) = 10$,
- for values of x in $(0, 1)$, $\log_2(x)$ is negative ($x = 2^y < 1$ only when $y < 0$),
- for x near 0 (and positive), $|\log_2(x)|$ is large.

The case when the base b is less than 1 is treated in Exercise 54.

EXERCISES for Section 1.2 *Key:* R–routine, M–moderate, C–challenging

1.[R] Graph the power function $x^{3/2}$, $x \geq 0$, and its inverse.

2.[R] Graph the power function x^5 and its inverse.

3.[R] Explain your calculator's response when you try to calculate $\log_{10}(-3)$.

4.[R] Explain your calculator's response when you try to calculate $\arcsin(2)$.

5.[R]

(a) Graph 2^x and $(1/2)^x$ on the same axes.

(b) How could you obtain the second graph from the first?

6.[R]

(a) Graph 3^x and $(1/3)^x$ on the same axes.

(b) How could you obtain the second graph from the first?

7.[R] For any base b, $b^0 = 1$. What is the corresponding property of logarithms? Explain.

8.[R] For any base b, $b^{x+y} = b^x b^y$. What is the corresponding property of logarithms? Explain. NOTE: If you have trouble with this exercise, study Appendix D.

9.[R] Explain why $\log_b(1/x) = -\log_b(x)$. ("The log of the reciprocal of x is the negative of the log of x.")

10.[R] Explain why $\log_b(c^x) = x\log_b(c)$. ("The log of a number raised to a power x is x times the log of the number.")

11.[R]

(a) Evaluate $\log_2(x)$ and $\log_4(x)$ at $x = 1, 2, 4, 8, 16$, and $1/16$.

(b) Graph $\log_2(x)$ and $\log_4(x)$ on the same axes (clearly label each point found in (a)).

(c) Compute $\frac{\log_4(x)}{\log_2(x)}$ for the six values of x in (a).

(d) Explain the phenomenon observed in (c).

(e) How would you obtain the graph of $\log_4(x)$ from that for $\log_2(x)$?

12.[R]

(a) Evaluate $\log_2(x)$ and $\log_8(x)$ at x=1, 2, 4, 8, 16, and 1/8.

(b) Graph $\log_2(x)$ and $\log_8(x)$ on the same axes (clearly label each point found in (a)).

(c) Compute $\frac{\log_8(x)}{\log_2(x)}$ for the six values of x in (a).

(d) Explain the phenomenon observed in (c).

(e) How would you obtain the graph of $\log_8(x)$ from that for $\log_2(x)$?

13.[R] Evaluate

(a) $\log_{10}(1000)$

(b) $\log_{100}(10)$

(c) $\log_{10}(0.01)$

(d) $\log_{10}(\sqrt{10})$

(e) $\log_{10}(10)$

14.[R] Evaluate

(a) $\log_3(3^{17})$

(b) $\log_3(1/9)$

(c) $\log_3(1)$

(d) $\log_3(\sqrt{3})$

(f) $\log_3(81)$

15.[R] Evaluate $5^{\log_5(17)}$. $3^{-\log_3(21)}$.

16.[R] Evaluate

17.[R] For positive x near 0, what happens to the functions 2^x, x^2 and $\log_2(x)$?

18.[R] For very large values of x what happens to the quotent $x^2/2^x$? Illustrate by using specific values for x.

19.[R] What happens to $(\log_2(x))/x$ for large values of x? Illustrate by citing specific x.

20.[R] Draw graphs of $\cos(x)$ for x in $[0, \pi]$, and $\arccos(x)$ on the same axes.

21.[R] Draw graphs of $\tan(x)$ for x in $(-\pi/2, \pi/2)$, and $\arctan(x)$ on the same axes.

In Exercises 22 to 38 evaluate the given expressions.

22.[R]

(a) $\sin^{-1}(1/2)$

(b) $\arcsin(1)$

(c) $\mathsf{inv}\sin(-\sqrt{3}/2)$

(d) $\arcsin(\sqrt{2}/2)$

23.[R]

(a) $\cos^{-1}(0)$

(b) $\mathsf{inv}\cos(-1)$

(c) $\arccos(1/2)$

(d) $\arccos(-1/\sqrt{2})$

24.[R]

(a) $\arctan(1)$

(b) $\mathsf{inv}\tan(-1)$

(c) $\arctan(\sqrt{3})$

(d) $\arctan(1000)$ (approximately)

25.[R]

(a) $\operatorname{arcsec}(2)$

(b) $\mathsf{inv}\sec(-2)$

(c) $\operatorname{arcsec}(\sqrt{2})$

(d) $\sec^{-1}(1000)$ (approximately)

26.[R]

(a) $\arcsin(0.3)$

(b) $\arccos(0.3)$

(c) $\arctan(0.3)$

(d) $\frac{\arcsin(0.3)}{\arccos(0.3)}$

NOTE: Observe that (c) and (d) are different.

27.[R] $\sin(\tan^{-1}(2))$.

28.[R] $\sin(\cos^{-1}(0.4))$.

29.[R] $\tan(\tan^{-1}(3))$.

30.[R] $\tan(\sin^{-1}(0.7))$.

31.[R] $\tan(\sec^{-1}(3))$.

32.[R] $\sec(\tan^{-1}(0.3))$.

33.[R] $\sin(\sec^{-1}(5))$.

34.[R] $\sec(\cos^{-1}(0.2))$.

35.[R] $\arctan(\tan(\frac{\pi}{3}))$.

36.[R] $\arcsin(\sin(\frac{-3\pi}{4}))$.

37.[R] $\arccos(\cos(\frac{5\pi}{2}))$.

38.[R] $\operatorname{arcsec}(\sec(\frac{-\pi}{3}))$.

In Exercises 39 to 42, use properties of logarithms to express $\log_{10} f(x)$ as simple as possible.

39.[M] $f(x) = \frac{(\cos(x))^7\sqrt{(x^2+5)^3}}{4+(\tan(x))^2}$

40.[M] $f(x) = \sqrt{(1+x^2)^5(3+x)^4\sqrt{1+2x}}$

41.[M] $f(x) = (x\sqrt{2+\cos(x)})^{x^2}$

42.[M] $f(x) = \sqrt{\frac{x(1+x)}{\sqrt{1+2x^3}}}$

43.[M] Imagine that your calculator fell on the floor and its multiplication and division keys stopped working. However, all the other keys, including the trigonometric, arithmetic, logarithmic, and exponential keys still functioned. Show how you would use your calculator to calculate the product and quotient of two positive numbers, a and b.

44.[M] (Richter Scale) In 1989, San Francisco and vicinity was struck by an earthquake that measured 7.1 on the **Richter scale**. The strongest earthquake in recent years had a Richter measure of 8.9 (Colombia-Equador in 1906 and Japan in 1933). A "major earthquake" typically has a measure of at least 7.5.
In his *Introduction to the Theory of Seismology*, Cambridge, 1965, pp. 271–272, K. E. Bullen explains the Richter scale as follows:
"Gutenburg and Richter sought to connect the magnitude M with the energy E of an earthquake by the formula

$$aM = \log_{10}\left(\frac{E}{E_0}\right)$$

and after several revisions arrived in 1956 at the result $a = 1.5$, $E_0 = 2.5 \times 10^{11}$ ergs." NOTE: Energy E is measured in ergs. M is the number assigned to the earthquake on the Richter scale. E_0 is the energy of the smallest instrumentally recorded earthquake.

(a) Deduce that $\log_{10}(E) \approx 11.4 + 1.5M$.

(b) What is the ratio between the energy of the earthquake that struck Japan in 1933 ($M = 8.9$) and the San Francisco earthquake of 1989 ($M = 7.1$)?

(c) What is the ratio between the energy of the San Francisco earthquake of 1906 ($M = 8.3$) and that of the San Francisco earthquake of 1989 ($M = 7.1$)?

(d) Find a formula for E in terms of M.

(e) If one earthquake has a Richter measure 1 larger than that of another earthquake, what is the ratio of their energies?

(f) What is the Richter measure of a 10-megaton H-bomb, that is, of an H-bomb whose energy is equivalent to that of 10 millon tons of TNT?

NOTE: One ton of TNT releases an energy of 4.2×10^6 ergs.

45.[M] Translate the sentence, "She has a five-figure annual income" into logarithms. How small can the income be? How large?

46.[M] As of 2006 the largest known prime was $2^{30402457} - 1$.

(a) When written in decimal notation, how many digits will it have?

(b) How many pages of this book would be needed to print it? (One page can hold about 6,400 digits.)

NOTE: There is a prize of $250,000 for the discovery of the first billion-digit prime. Do a Google search for "largest prime".

47.[M]

(a) In many calculators the log key refers to base-ten logarithms. You can use it to find logarithms to any base $b > 0$. To see why, start with the equation $b^{\log_b(x)} = x$ and then take $\log_{10}$ of both sides. This gives the formula

$$\log_b(x) = \frac{\log_{10}(x)}{\log_{10}(b)}.$$

(b) Use (a) to find $\log_3(7)$. (Why should the result be between 1 and 2?)

(Semi-log graphs) In most graphs the scale on the y-axis is the same as the scale on the x-axis, or a constant multiple of it. However, to graph a rapidly increasing function, such as 10^x, it is convenient to "distort" the y-axis. Instead of plotting the point (x, y) at a height of, say, y inches, you plot it at a height of $\log_{10} y$ inches. So the datum $(x, 1)$ could be drawn with height zero, the datum $(x, 10)$, would have height 1, and the datum $(x, 100)$ would have height 2 inches. Instead of graphing $y = f(x)$, you graph $Y = \log_{10} f(x)$. In particular, if $f(x) = 10^x$, $y = \log_{10} 10^x = x$: the graph would be a straight line. To avoid having to calculate a bunch of logarithms, it is convenient to use semi-log graph paper, shown in Figure 1.2.13.

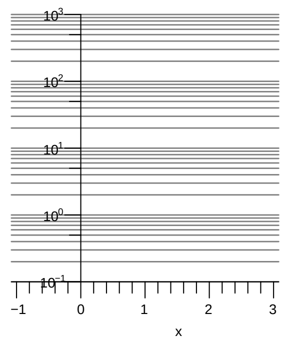


Figure 1.2.13:

48.[C] Using semi-log paper, graph $y = 2 \cdot 3^x$.

49.[C] Using semi-log paper, graph $y = \frac{2}{3^x}$.

50.[C] (Slide Rule) This exercise shows how to build a slide rule by exploiting the equation $\log_b(xy) = \log_b(x) + \log_b(y)$. We will use $\log_2$ for convenience.

Step 1. Mark on the bottom edge of a stick (or page) the numbers 2^0, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, and $2^4 = 16$, placing 2^n at a distance n cm from the left end. In other words, place each number x, at a distance $\log_2(x)$ cm from the left edge. Figure 1.2.14 shows only numbers with convenient integer logarithms, with base 2.

Step 2. Do the same thing as the top edge of another stick or sheet of paper.

Step 3. You now have a slide rule. To compute 4×8, say, with your slide rule, slide the bottom stick along the top stick until its left edge is next to the 4 of the top stick. The product 4×8 appears above the 8 on the lower stick. Why?

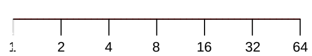


Figure 1.2.14:

51.[C] Newton computed the logarithms of 0.8, 0.9, 1.1, and 1.2 to 57 decimal places by hand using a method that you will learn about in Section 10.4.

(a) Show how to compute $\log(2)$, using $\log(1.2)$, $\log(0.8)$ and $\log(0.9)$.

(b) Show how to compute $\log(3)$, using $\log(2)$, $\log(1.2)$ and $\log(0.8)$.

(c) Show how to compute $\log(4)$, using $\log(2)$.

(d) Show how to compute $\log(5)$, using $\log(2)$ and $\log(0.8)$.

(e) How would you then compute $\log(6$, $\log(8)$, $\log(9)$, and $\log(10)$.

(f) How would you then estimate $\log(11)$.

NOTE: You don't need to know the base. Why?

52.[M] The graph of $y = \log_2(x)$ consists of the part to the right of $(1, 0)$ and the part to the left of $(1, 0)$. Are the two parts congruent?

53.[C] Say that you have drawn the graph of $y = \log_2(x)$. Jane says that to get the graph of $y = \log_2(4x)$, you just raise that graph 2 units parallel to the y-axis. Sam says, "No, just shrink the x-coordinate of each point on the graph by a factor of 4." Who is right?

54.[C] Answer the following questions about $y = \log_b(x)$ where $0 < b < 1$.

(a) Sketch the graphs of $y = b^x$ and $y = \log_b(x)$ on the same set of axes.

(b) What is the domain of $\log_b$?

(c) What is the x-intercept? That is, solve $\log_b(x) = 0$.

(d) For what values of x is $\log_b(x)$ positive? negative?

(e) Is the graph of $y = \log_b(x)$ an increasing or decreasing function?

(f) What can you say about the values of $\log_b(x)$ when x is close to zero (and in the domain)?

(g) What can you say about the values of $\log_b(x)$ when x is a large positive number?

(h) What can you say about the values of $\log_b(x)$ when x is a large negative number?

55.[C] Let a, b, c, d be constants such that $ad - bc \neq 0$.

(a) Show that $y = (ax + b)/(cx + d)$ is one-to-one.

(b) For which a, b, c, d does the function in (a) equal its inverse function?

56.[C] Prove that $\log_3(2)$ is irrational, that is, not rational. HINT: Assume that it is rational, that is, equal to m/n for some integers m and n, and obtain a contradiction.

1.3 Building More Functions from Basic Functions

In this section we complete the list of functions needed for calculus. Our starting point is the basic functions introduced in Section 1.1. We will use just two methods to build more complicated functions from x^k, b^x, $\sin(x)$, $\cos(x)$, $\tan(x)$, and their inverses. For instance we will see how to obtain

$$f(x) = \frac{\sin(2x) + 3 + 4x + 5x^2}{\log_2(x) + 3^{-5x} + \sqrt{1 + x^3}}. \tag{1.3.1}$$

Before we go into the details of how we construct new functions from old ones, we must introduce one more type of basic function. These functions are so simple, however, that they did not deserve to appear with the functions in the preceding section. They are the constant functions, whose graphs are horizontal lines. (See Figure 1.3.1.)

The Constant Functions

DEFINITION *(Constant Function)* A function $f(x)$ is **constant** if there is a number C such that $f(x) = C$ for all x in its domain. A special constant function is the **zero function**: $f(x) = 0$.

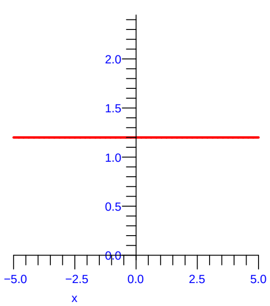


Figure 1.3.1:

Using the Four Arithmetic Operations: $+$, $-$, $\times$, $/$

Given two functions f and g, we can produce other functions from them by using the four operations of arithmetic:

$f + g$: for an input value x, the function assigns $f(x) + g(x)$ as the output

$f - g$: for an input value x, the function assigns $f(x) - g(x)$ as the output

fg: for an input value x, the function assigns $f(x)g(x)$ as the output

f/g: for an input value x with $g(x) \neq 0$, the function assigns $f(x)/g(x)$ as the output

The domains of $f+g$, $f-g$, and fg consist of the numbers that belong to both the domain of f and the domain of g. The domain of f/g is a little different because division by zero is not defined. The function f/g is defined for all numbers x that belong to the domain of f and the domain of g with the extra condition that $g(x) \neq 0$.

With the aid of these constructions we can build any polynomial or rational function from the simple function $f(x) = x$, called the **identity function**, and the constant functions.

A **polynomial** is a function of the form $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where the coefficients a_0, a_1, a_2, $\ldots$, a_n are numbers. If a_n is not zero, the **degree** of the polynomial is n. A **rational function** is the quotient of two polynomials. The domain of a polynomial is the set of all real numbers. The domain of a rational polynomial is all real numbers except those where the denominator is zero.

EXAMPLE 1 Use addition, subtraction, and multiplication to form the polynomial $F(x) = x^3 + 3x - 7$.

SOLUTION We first build each of the three terms: x^3, $3x$, and 7. The last of these is just a constant function. Multiplying the identity function x and the constant function 3 gives $3x$. The first term is obtained by first multiplying x and x to obtain x^2. Then multiplying x^2 and x yields x^3. Adding x^3 and $3x$ gives $x^3 + 3x$. Lastly, subtract the constant function 7 to obtain $x^3 + 3x - 7$.

Notice that each of the three functions involved in forming F is defined for all real numbers. As a result, the domain of F is also all real numbers, $(-\infty, \infty)$. ◇

Example 1 shows how to build any polynomial using $+$, $-$, and $\times$. Constructing rational functions also requires one use of the division operator.

But how would we build a function like $\sqrt{1+3x}$? This leads us to the most important technique for combining two or more functions to build more complicated functions.

Composite Functions

Given two functions f and g we can use the output of g as the input for f. That is, we can find $f(g(x))$. For instance, if $g(x) = 1 + 3x$ and f is the square root function, $f(x) = \sqrt{x}$, then $f(g(x)) = f(1 + 3x) = \sqrt{1+3x}$. This brings us to the definition of a composite function.

> **DEFINITION** *(Composition of functions)* Let X, Y, and Z be sets. Let g be a function from X to Y and let f be a function from Y to Z. Then the function that assigns to each element x in X the element $f(g(x))$ in Z is called the **composition** of f with g. It is denoted $f \circ g$, which is read as "f circle g" or as "f composed with g".

Some or all of the sets X, Y, and Z could be the same set.

Thinking of f and g as input-output machines we may consider $f \circ g$ as the machine built by hooking up the machine for f to process the outputs of the machine for g (see Figure 1.3.2).

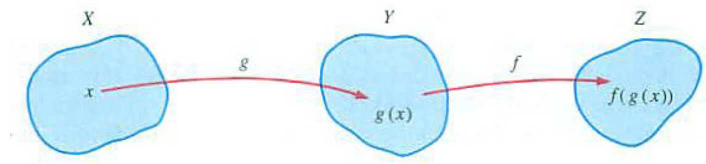


Figure 1.3.2: The output of the g machine, $g(x)$, becomes the input for the f machine. The result is the composition of f with g, $(f \circ g)(x) = f(g(x))$.

Most functions we encounter are composite functions. For instance, $\sin(2x)$ is the composition of $g(x) = 2x$ and $f(x) = \sin(x)$. Of course, we can hook up three or more functions to make even fancier functions. Consider $\sin^3(2x) = (\sin(2x))^3$. This function is built up as follows:

$$x \longrightarrow 2x \longrightarrow \sin(2x) \longrightarrow (\sin(2x))^3. \tag{1.3.2}$$

It is the composition of three functions: the first doubles the input, the second takes the sine of its input, and the third cubes its input.

The order does matter. If, instead, you cube first, then take the sine, and then double the input you obtain:

$$x \longrightarrow x^3 \longrightarrow \sin(x^3) \longrightarrow 2\sin(x^3). \tag{1.3.3}$$

When you enter a function on your calculator or on a computer, you have to be careful of the order in which the functions are applied as you evaluate a composite function. The specific way that you would evaluate $\sin(\log_{10}(240))$ depends on your calculator. On a traditional scientific calculator you enter 240 followed by the log10 key, and finally the sin key. On many of the newer graphing calculators you would press the sin key followed by the log10 key, then 240, followed by two right parentheses,)), and, finally, the Enter key. Note that these two approaches are different. If you press the sin key before log10, you will get $\log_{10}(\sin 240)$. For most computer software it is necessary to use parentheses to indicate inputs to functions. In this case you might enter sin(log10(240)).

Before pressing the sin key, be sure to check that your calculator is in radians mode.

To describe the build-up of a composite function it is convenient to use various letters, not just x, to denote the variables. This is illustrated in Examples 2 to 4.

If your calculator is in degree mode, you will find that $\sin(240°) < 0$ and so $\log_{10}(\sin(240°))$ is not defined.

EXAMPLE 2 Show how the function $\sqrt{4 - x^2}$ is built up by the composition of functions. Find its domain.

SOLUTION The function $\sqrt{4 - x^2}$ is obtained by applying the square-root function to the function $4 - x^2$. To be specific, let

$$g(x) = 4 - x^2 \qquad \text{and} \qquad f(u) = \sqrt{u} \ (u \geq 0). \tag{1.3.4}$$

Then

$$f(g(x)) = f(4 - x^2) = \sqrt{4 - x^2}. \tag{1.3.5}$$

The square-root function is defined for all $u \geq 0$ and the polynomial $g(x)$ is defined for all numbers. So $f(g(x))$ is defined only when $g(x) \geq 0$:

$$\begin{aligned} g(x) &\geq 0 \\ 4 - x^2 &\geq 0 \\ 4 &\geq x^2 \\ 2 &\geq |x|. \end{aligned}$$

Thus, the domain of $\sqrt{4 - x^2}$ is the closed interval $[-2, 2]$. $\diamond$

EXAMPLE 3 Express $1/\sqrt{1 + x^2}$ as a composition of three functions. Find the domain of this function.

SOLUTION Call the input x. First, we compute $1 + x^2$. Second, we take the square root of that output, getting $\sqrt{1 + x^2}$. Third, we take the reciprocal of that result, getting $1/\sqrt{1 + x^2}$. In summary, we form

$$u = 1 + x^2, \qquad \text{then } v = \sqrt{u} \qquad \text{then } y = \frac{1}{v}. \tag{1.3.6}$$

Given x, we first evaluate the polynomial $1 + x^2$, then apply the square-root function, then the reciprocal function.

The domain of a polynomial consists of all real numbers; the domain of the square-root function is $v \geq 0$; and the domain of the reciprocal function is all numbers except zero. Because $u = 1 + x^2 \geq 1$, $v = \sqrt{u} = \sqrt{1 + x^2}$ is defined for all x. Moreover, $v = \sqrt{1 + x^2} \geq 1$, so that $y = \frac{1}{v} = 1/\sqrt{1 + x^2}$ is defined for all real numbers x. $\diamond$ The function in Example 3 can also be written as the composition of two functions: $x \longrightarrow 1 + x^2 \longrightarrow (1 + x^2)^{-1/2}$.

EXAMPLE 4 Let f be the cubing function and g the cube-root function. Compute $(f \circ g)(x)$, $(f \circ f)(x)$ and $(g \circ f)(x)$.

SOLUTION In terms of formulas, $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$.

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) = f(\sqrt[3]{x}) = \left(\sqrt[3]{x}\right)^3 = x. && (1.3.7) \\ (f \circ f)(x) &= f(f(x)) = f(x^3) = \left(x^3\right)^3 = x^9. && (1.3.8) \\ (g \circ f)(x) &= g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x. && (1.3.9) \end{aligned}$$

Observe that the domains of f and g are $(-\infty, \infty)$. Therefore, each of $f \circ g$, $f \circ f$, and $g \circ f$ is defined for all real numbers.

Notice that both $f \circ g$ and $g \circ f$ are the identity function. Whenever g is the inverse of f, $f \circ g$ and $g \circ f$ are the identity function. Each function undoes the action of the other. ◇

EXAMPLE 5 Give two different ways of obtaining the function $1/f(x)$ from the function f.

SOLUTION The first approach is to view $1/f(x)$ as the quotient of the constant function 1 and the function $f(x)$.

This function can also be viewed as a composition. The quotient $1/f(x)$ can be obtained in two steps: First, evaluate $f(x)$. Second, take the reciprocal of the result. So, if $g(x) = 1/x$, then

$$\frac{1}{f(x)} = g(f(x)). \tag{1.3.10}$$

Regardless of the way in which $1/f(x)$ is constructed, the domain is all real numbers for which $f(x) \neq 0$. ◇

EXAMPLE 6 Show that every power function x^k, $x > 0$, can be constructed as a composition using exponential or logarithmic functions.

SOLUTION The first step is to write $x = 2^{\log_2(x)}$. Then, $x^k = \left(2^{\log_2(x)}\right)^k$ or, by properties of exponentials, $x^k = 2^{k \log_2(x)}$. So x^k is the composition of three functions: First, find $\log_2(x)$, then multiply by the constant function k, and then raise 2 to the resulting power. ◇

That a power function can be expressed in terms of an exponential function will be used in Chapter 4.

OBSERVATION *(Consequences of Example 6)*

1. The construction in Example 6 provides a meaning to functions like $x^{\sqrt{2}}$ and $x^{-\pi}$ for $x > 0$.
2. As a result of Example 6 we could remove the power functions from our list of basic functions in Section 1.1. We choose not to do so because power functions with integer exponents are so common and in many instances we want to define a power function for all inputs (not just positive numbers).
3. It might seem surprising that the power functions can be expressed in terms of exponentials (and logarithms). An even more astonishing result is that trigonometric functions, such as $\sin(x)$, can also be expressed in terms of exponentials, as shown in Section 12.7.

Summary

This section showed how to build more complicated functions from power, exponential, and trigonometric functions and their inverses, and the constant functions. One method is to simply add, multiply, subtract, or divide outputs. The other method is the "composition of functions" in which one function operates on the output of a second function. Composite functions are extremely important, especially when we calculate derivatives beginning in Chapter 3.

> **WARNING** *(Traveler's Advisory about Notation)* Be careful when composing functions when one of them is a trigonometric function. For instance, what is meant by $\sin x^3$? Is it $\sin(x^3)$ or $(\sin(x))^3$? Do we first cube x, then take the sine, or the other way around? There is a general agreement that $\sin x^3$ stands for $\sin(x^3)$; you cube first, then take the sine.
>
> Spoken aloud, $\sin x^3$ is usually "the sine of x cubed," which is ambiguous. We can either insert a brief pause – "sine of (pause) x cubed" – to emphasize that x is cubed rather than $\sin(x)$, or rephrase it as "sine of the quantity x cubed."
>
> On the other hand $(\sin(x))^3$, which is by convention usually written as $\sin^3(x)$, is spoken aloud as "the cube of $\sin(x)$" or "sine cubed of x."
>
> Similar warnings apply to other trigonometric functions and logarithmic functions.

EXERCISES for Section 1.3 *Key:* R–routine, M–moderate, C–challenging

The function $y = \sqrt{1+x^2}$ is the composition of $s = 1 + x^2$ and $y = \sqrt{s}$. In Exercises 1 to 12 use a similar format to build the given functions as the composition of two or more functions.

1.[R] $\sin(2x)$

2.[R] $\sin^3(x)$

3.[R] $\sin(3x)$

4.[R] $\sin(x^3)$

5.[R] $\sin^2(x^3)$

6.[R] 2^{x^2}

7.[R] $(x^2+3)^{10}$

8.[R] $\log_{10}(1+x^2)$

9.[R] $1/(x^2+1)$

10.[R]

$\cos^3(2x+3)$

11.[R] $\left(\frac{2}{3x+5}\right)^3$

12.[R] $\arcsin(\sqrt{x})$

13.[R] These tables show some of the values of functions f and g:

x	1	2	3	4	5
$f(x)$	6	8	9	7	10

x	6	7	8	9	10
$g(x)$	4	3	2	5	1

(a) Find $f(g(7))$.

(b) Find $g(f(3))$.

14.[R] Figure 1.3.3 shows the graphs of functions f and g.

(a) Estimate $f(g(0.6))$.

(b) Estimate $f(g(0.3))$.

(c) Estimate $f(f(0.5))$.

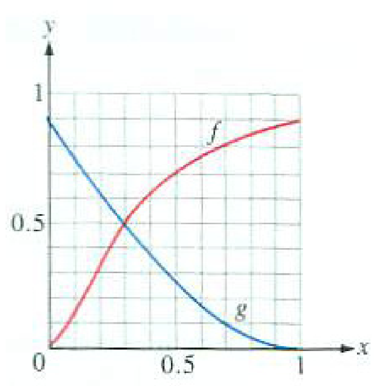


Figure 1.3.3:

In Exercises 15 and 24 write y as a function of x.

15.[R] $u = \sin(x)$, $y = u^2$

16.[R] $u = x^3$, $y = 1/u$

17.[R] $u = 2x^2 - 3$, $y = 1/u$

18.[R] $u = \sqrt{x}$, $y = u^2$

19.[R] $u = \sqrt{x}$, $y = \sin(u)$

20.[R] $u = x^2$, $y = 2^u$

21.[R] $v = 2x$, $u = v^2 - 1$, $y = u^2$

22.[R] $v = \sqrt{x}$, $u = 1+v$, $y = u^2$

23.[R] $v = x + x^2$, $u = \sin(v)$, $y = u^3$

24.[R] $v = \tan(x)$, $u = 1 + v^2$, $y = \cos(u)$

25.[M] Let $f(x) = 2x^2 - 1$ and $g(x) = 4x^3 - 3x$. Show that $(f \circ g)(x) = (g \circ f)(x)$. [Rare indeed are pairs of polynomials that commute with each other under composition, as you may convince yourself by trying to find more examples.] NOTE: Of course, any two powers, such as x^3 and x^4, commute. (The composition of x^3 and x^4 in either order is x^{12}, as may be checked.)

26.[M] Let $f(x) = 1/(1-x)$. What is the domain of f? of $f \circ f$? of $f \circ f \circ f$? Show that $(f \circ f \circ f)(x) = x$ for all x in the domain of $f \circ f \circ f$.

27.[M] Let $g(x) = x^2$. Find all first degree polynomials $f(x) = ax + b$, $a \neq 0$, such that $f \circ g = g \circ f$, that is, $f(g(x)) = g(f(x))$.

28.[M] Let $f(x) = x^3$. Is there a function $g(x)$ such that $(f \circ g)(x) = x$ for all numbers x? If so, how many such functions are there?

29.[M] Let $f(x) = x^4$. Is there a function $g(x)$ such that $(f \circ g)(x) = x$ for all negative numbers x? If so, how many such functions are there?

30.[M] Let $f(x) = x^4$. Is there a function $g(x)$ such that $(f \circ g)(x) = x$ for all positive numbers x? If so, how many such functions are there?

31.[M] Figure 1.3.4 shows the graph of a function f whose domain is $[0,1]$. Let $g(x) = f(2x)$.

(a) What is the domain of g?

(b) Graph $y = g(x)$

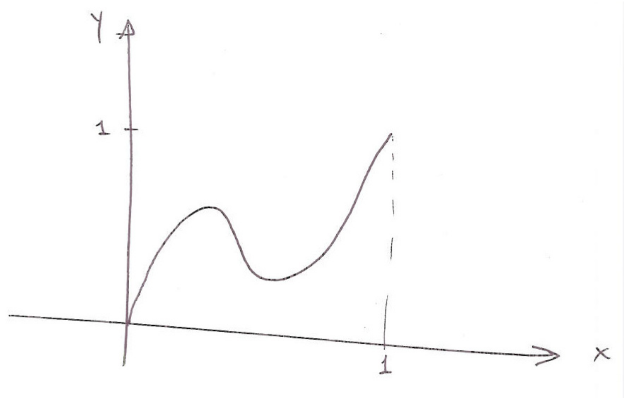


Figure 1.3.4:

32.[M] Show that there is a function $u(x)$ such that $\cos x = \sin u(x)$. NOTE: This shows that we didn't need to include $\cos x$ among our basic functions.

33.[M] Find a function $u(x)$ such that $3^x = 2^{u(x)}$.

34.[C] If f and g are one-to-one, must $f \circ g$ be one-to-one?

35.[C] If $f \circ g$ is one-to-one, must f be one-to-one? Must g be one-to-one?

36.[C] If f has an inverse, $\mathsf{inv}f$, compute $(f \circ \mathsf{inv}f)(x)$ and $((\mathsf{inv}f) \circ f)(x)$.

37.[C] Let $g(x) = x^2$. Find all second-degree polynomials $f(x) = ax^2+bx+c$, $a \neq 0$, such that $f \circ g = g \circ f$, that is, $f(g(x)) = g(f(x))$.

38.[C] Let $f(x) = 2x + 3$. Find all functions of the form $g(x) = ax + b$, a and b constants, such that $f \circ g = g \circ f$.

39.[C] Let $f(x) = 2x + 3$. Find all functions of the form $g(x) = ax^2 + bx + c$, a, b, and c constants, such that $f \circ g = g \circ f$.

40.[C] Find all functions of the form $f(x) = 1/(ax + b)$, $a \neq 0$, such that $(f \circ f \circ f)(x) = x$ for all x in the domain of $f \circ f \circ f$.

41.[C] (Induction) This exercise rests on the identifies $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$, $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$, and $\cos^2 x + \sin^2 x = 1$.

(a) Show that $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = 2\cos^2(x) - 1$.

(b) Show that $\sin(3x) = 3\sin(x) - 4\sin^3(x)$ and $\cos(3x) = 4\cos 3(x) - 3\cos(x)$.

(c) Show that $\sin(4x) = \cos(x)(4\sin(x) - 8\sin^3(x))$ and $\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$.

(d) Use induction to show that for each positive integer n, $\cos(nx)$ is a polynomial in $\cos(x)$. That is, there is a polynomial P_n such that $\cos(nx) = P_n(\cos(x))$. NOTE: You will have to consider the form of $\sin(nx)$, n odd or even, in the induction.

(e) Explain why $P_n \circ P_m = P_m \circ P_n$. NOTE: This does not require the explicit formulas for P_n and P_m.

1.4 Geometric Series

Let a and r be numbers. The sequence of number

$$a, ar, ar^2, ar^3, \ldots$$

is called a **geometric sequence**. Its first **term** is a. Each term after the first term is obtained by multiplying its predecessor by r, which is called the **ratio**. The n^{th} term is ar^{n-1}.

A finite collection of consecutive terms from a geometric sequence is also called a **geometric progression**.

Let S_n be the sum of the first n terms of the geometric sequence:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

There is a short formula for this sum, which we will use several times.

To find this formula, subtract rS_n from S_n, as follows:

$$\begin{array}{rcl} S_n & = & a + \quad\quad ar + ar^2 + \cdots + ar^{n-1} \\ rS_n & = & \quad\quad ar + ar^2 + \cdots + ar^{n-1} + ar^n. \end{array}$$

Because of the many cancellations,

$$S_n - rS_n = a - ar^n.$$

If r is not 1, we can divide by $1 - r$ to obtain:

Short Formula for the Sum of a Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1$$

EXAMPLE 1 Find (a) $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32}$ and (b) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81}$.

SOLUTION (a) Here $a = 3$, $r = \frac{1}{2}$, and $n = 6$. The sum is

$$S_6 = \frac{3\left(1-\frac{1}{2}\right)^6}{1-\frac{1}{2}} = 6\left(1-\left(\frac{1}{2}\right)^6\right) = \frac{378}{64} = \frac{189}{32}.$$

(b) In this case $a = 1$, $r = \frac{-1}{3}$, and $n = 5$. So the sum is

$$S_5 = \frac{1\left(1-\frac{-1}{3}\right)^5}{1-\frac{-1}{3}} = \frac{1-\left(\frac{-1}{3}\right)^5}{\frac{4}{3}} = \frac{3}{4}\left(1+\left(\frac{1}{3}\right)^5\right) = \frac{61}{81}.$$

◇

For a positive ratio r less than 1, the Figure 1.4.1 provides a geometric way to sum $1+r+r^2+\cdots+r^{n-1}$. The points with coordinates r, r^2, r^3, …, r^n cut the interval $[r^n, 1]$ into n intervals. The sum of the lengths of these intervals is the length of the interval $[r^n, 1]$, which is $1-r^n$. Thus

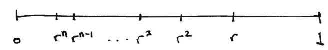


Figure 1.4.1:

$$1-r^n = (1-r)+(r-r^2)+(r^2-r^3)+\cdots+(r^{n-1}-r^n)$$

Notice that $(1-r)$ can be factored from each of the terms on the right-hand side of this equation. So

$$\begin{aligned} 1-r^n &= (1-r)+(1-r)r+(1-r)r^2+\cdots+(1-r)r^{n-1} \\ &= (1-r)(1+r+r^2+\cdots+r^{n-1}). \end{aligned}$$

Because r is not 1, $1-r$ is not zero. It follows that

$$1+r+r^2+\cdots+r^{n-1} = \frac{1-r^n}{1-r}.$$

Let x and a be two numbers and consider the sequence

$$x^{n-1}, ax^{n-2}, a^2x^{n-3}, a^3x^{n-4}, \ldots, a^{n-3}x^2, a^{n-2}x, a^{n-1}. \tag{1.4.1}$$

The exponent of x decreases from $n-1$ to 0 and the exponent of a increases from 0 to $n-1$.

While it might not look like it at first, (1.4.1) is the first n terms of a geometric sequence. The first term is x^{n-1} and the ratio is a/x. Thus, assuming x is not 0 or a,

$$\begin{aligned} &x^{n-1}+ax^{n-2}+a^2x^{n-3}+a^3x^{n-4}+\cdots+a^{n-3}x^2+a^{n-2}x+a^{n-1} \\ = \; &x^{n-1}\left(\frac{\left(1-\left(\frac{a}{x}\right)^n\right)}{1-\frac{a}{x}}\right) = \frac{x^{n-1}\left(\frac{x^n-a^n}{x^n}\right)}{\frac{x-a}{x}} = \frac{x^n-a^n}{x-a}. \end{aligned}$$

This leads us to conclude that

$$\boxed{x^{n-1}+ax^{n-2}+a^2x^{n-3}+a^3x^{n-4}+\cdots+a^{n-3}x^2+a^{n-2}x+a^{n-1} = \frac{x^n-a^n}{x-a} \qquad x \neq a} \tag{1.4.2}$$

See page 67.

In Chapter 2 we will use (1.4.2) in the reverse order, to express the quotient $\frac{x^n-a^n}{x-a}$ as a sum of n terms.

Equation (1.4.2) can also be established by considering the factorizations of x^n-a^n:

$$\begin{aligned} x^2-a^2 &= (x-a)(x+a) \\ x^3-a^3 &= (x-a)(x^2+ax+a^2) \\ x^4-a^4 &= (x-a)(x^3+ax^2+a^2x+a^3) \qquad (1.4.3) \\ & \qquad (1.4.4) \end{aligned}$$

and so on. To establish (1.4.3), for instance, multiply out its right-hand side:

$$\begin{aligned}(x-a)(x^3+ax^2+a^2x+a^3) &= (x^4+ax^3+a^2x^2+a^3x)-(ax^3+a^2x^2+a^3x+a^4)\\ &= x^4-a^4.\end{aligned}$$

Summary

The key idea of this section is that the sum of the n numbers $a+ar+ar^2+\cdots+ar^{n-1}$ equals $a\frac{1-r^n}{1-r}$ so long as r is not 1. If r is 1, then the sum is just na, because each summand is a.

Finite Geometric Series

Let b and r be numbers and n a positive integer. An expression of the form

$$b+br+br^2+\cdots+br^n \tag{1.4.5}$$

is called a **finite geometric series**. For instance, when $b=1$ and $r=x$, it takes the form

$$1+x+x^2+\cdots+x^n,$$

the Maclaurin polynomial of order n associated with $1/(1-x)$.
If $b=x^n$ and $r=a/x$, (1.4.5) becomes

$$x^n+x^n\left(\frac{a}{x}\right)+x^n\left(\frac{a}{x}\right)^2+\cdots+x^n\left(\frac{a}{x}\right)^n,$$

which reduces to

$$x^n+ax^{n-1}+a^2x^{n-2}+\cdots+a^n.$$

We will encounter this shortly, in Section 2.2, when weneed to factor x^n-a^n.
It is easy to find a short formula for the sum in (1.4.5), which we call S_n. We have

$$\begin{array}{rrcccccccccl} & S_n & = & b & + & br & + & br^2 & + & \cdots & + & br^n \\ \text{and} & rS_n & = & & & br & + & br^2 & + & \cdots & + & br^n+br^{n+1}.\end{array}$$

Subtracting rS_n from S_n yields

$$(1-r)S_n=b-br^{n+1},$$

and we have, if r is not 1,

$$S_n=\frac{b\left(1-r^{n+1}\right)}{1-r}.$$

This result will be used extensively later, particularly in Section 5.4.

EXERCISES for Section 1.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 6 calculate the sum using the formula for the sum of a geometric progression.

1.[R] $1+3+9+27+81+243$

2.[R] $1-3+9-27-81+243$

3.[R] $2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$

4.[R] $0.5 - 0.05 + 0.005 - 0.0005 + 0.00005 - 0.000005 + 0.0000005$

5.[R] $a^4 + a^3b + a^2b^2 + ab^3 + b^4$

6.[R] $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$

In Exercises 7 and 8 write the given polynomial as a product of two polynomials

7.[R] $x^6 - a^6$

8.[R] $x^9 - a^9$

1.5 Logarithms

How many 2's must be multiplied to get 32? Whatever the answer is, it is called "the logarithm of 32 to the base 2." Because $2^5 = 32$, the logarithm of 32 to the base 2 is 5. More generally, a logarithm is defined in terms of an exponential function.

Definition of $\log_b(c)$

Definition of Logarithm to the Base b

Let b and c be positive numbers, $b \neq 1$. There is a number d such that

$$b^d = c.$$

The exponent d is called the **logarithm of c to the base b**. It is denoted

$$\log_b(c).$$

By the definition of a logarithm,

$$b^{\log_b(c)} = c.$$

EXAMPLE 1 Find (a) $\log_{10}(1000)$, (b) $\log_2(1024)$, (c) $\log_9(3)$, and (d) $\log_{16}\left(\frac{1}{4}\right)$.

SOLUTION (a) Because $10^3 = 1000$, $\log_{10}(1000) = 3$.

(b) Because $2^{10} = 1024$, $\log_2(1024) = 10$.

(c) Because $9^{1/2} = 3$, $\log_9(3) = \frac{1}{2}$.

(d) Because $16^{-1/2} = \frac{1}{4}$, $\log_{16}\left(\frac{1}{4}\right) = \frac{-1}{2}$. ◇

Every property of an exponential function can be translated into a property of logarithms. For instance, here is how we translate the equation $b^{x+y} = b^x b^y$ into logarithms.

Let $c = b^x$ and $d = b^y$. We have

$$x = \log_b(c) \qquad \text{and} \qquad y = \log_b(d). \tag{1.5.1}$$

Because

$$cd = b^x b^y = b^{x+y}$$

we know

$$\log_b(cd) = x + y.$$

Using (1.5.1), we conclude that

$$\log_b(cd) = \log_b(c) + \log_b(d).$$

This generalizes to the logarithm of the product of several numbers. In words,

The log of a product of two or more numbers is the sum of the logs of these numbers.

Table 1.5.1 lists the translation of properties of exponential functions into the terminology of logarithms.

Exponential Language			Logarithm Language		
b^{x+y}	$=$	$b^x b^y$	$\log_b cd$	$=$	$\log_b c + \log_b d$
b^0	$=$	1	$\log_b 1$	$=$	0
b^1	$=$	b	$\log_b b$	$=$	1
b^{-x}	$=$	$1/b^x$	$\log_b(1/c)$	$=$	$-\log_b c$
$(b^x)^y$	$=$	b^{xy}	$\log_b c^d$	$=$	$d \log_b c$

Table 1.5.1:

Figure 1.5.1 is the graph of $y = \log_2(x)$. Notice that as x increases, so does $\log_2(x)$, but very slowly. Also, when x is near 0, $\log_2(x)$ is negative but has large absolute values.

Logarithms are used to simplify products and quotients that involve powers. For instance,

$$\begin{aligned}\log_b\left(\frac{\sqrt{x}(2+x)^3}{(1+x^2)^5}\right) &= \log_b(\sqrt{x}) + \log_b\left((x+2)^3\right) - \log_b\left(\left(1+x^2\right)^5\right)\\ &= \frac{1}{2}\log_b(x) + 3\log_b(2+x) - 5\log_b\left(1+x^2\right).\end{aligned}$$

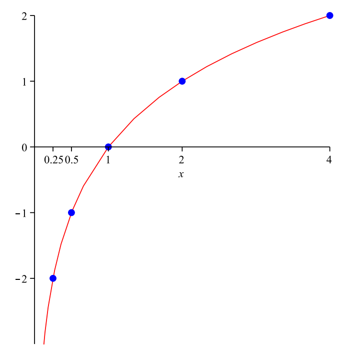


Figure 1.5.1:

In the final expression, most of the exponents and radical sign no longer appear. There is no way to simplify $\log_b(2+x)$ and $\log_b(1+x^2)$.

Summary

This section reviews logarithms, which are simply a different way of talking about exponents. The two key properties of logarithms for a positive base b are $\log_b(xy) = \log_b(x) + \log_b(y)$ and $\log_b(x^y) = y\log_b(x)$.

The word "logarithm" comes from the Greek. In a Greek restaurant, to get the bill, you ask the waiter for the "logarismo".

EXERCISES for Section 1.5 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 5 establish the given property of logarithms by using an appropriate property of exponentials. (Assume $b > 0$.)

1.[R] $\log_b(1) = 0$

2.[R] $\log_b(b) = 1$

3.[R] $\log_b(1/c) = -\log_b(c)$ $(c > 0)$

4.[R] $\log_b(c^d) = d\log_b(c)$ $(c > 0)$

5.[R] $\log_b(c/d) = \log_b(c) - \log_b(d)$ $(c > 0,\ d > 0)$

6.[R] Why is $\log_b(c)$ defined only for positive values of c?

7.[R]

(a) Graph $\log_{1/2}(x)$ and $\log_2(x)$.

(b) How is $\log_{1/b}(c)$ related to $\log_b(c)$?

8.[R] How is $\log_{b^2}(c)$ related to $\log_b(c)$?

9.[R] Evaluate (a) $\log_b\left(\sqrt{b}\right)$, (b) $\log_b\left(\frac{b^3}{\sqrt{b}}\right)$, (c) $\log_b\left(\sqrt{b}\sqrt[3]{b}b^4\right)$

10.[R] Simplify $\log_2\left(\frac{\left(x^3\right)^5\sqrt[3]{x+2}\left(1+x^2\right)^15}{x^5+7}\right)$.

11.[R] Show that $\frac{\log_b(x)-\log_b(y)}{c} = \log_b\left(\left(\frac{x}{y}\right)^{1/c}\right)$.

12.[R] What happens to $\log_{10}(x)/x$ for large values of x? Hint: Experiment and form a conjecture.

13.[R] Translate "She has a five-figure income" into logarithms.

14.[M] How would you find $\log_5(3^7)$ if your calculator has only a key for logarithms to the base ten? Hint: Start with the equation $5^x = 3^7$ and take logarithms to the base ten.

15.[M] Until the appearance of calculators, slide rules were commonly used for multiplication and division. Now, the International Slide Rule Museum (`http://www.sliderulemuseum.com/` is the world's largest digital repository of slide rule information. To see how the slide rule multiplies two numbers, mark two pieces of paper with the numbers 1, 2, 4, 8, 16, and 32 placed at equal distances apart, as shown in Figure **??**. To multiply, say, 4 times 8, slide the lower paper so its 1 is under the 4. Then the product of 4 and 8 appears above the 8.

(a) Why does the slide rule work?

(b) How would you make a slide rule for multiplying that has all the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 on both scales?

16.[M]

(a) Show that for positive numbers b and c, neither equal to 1, $\log_b(x)/\log_c(x)$ equals $\log_b(c)$, independent of x $(x > 0)$. Hint: Start with $b^{\log_b(x)} = x$.

(b) What does (a) imply about the graphs of $y = \log_b(x)$ and $y = \log_c(x)$?

17.[M] A calculator often has a key for logarithms to base ten, labeled log, and one labeled $\ln x$ for which the base is the number e. You will meet e in Section 2.2; it is approximately 2.718. Using only the log key (and +, -, *, and /), how would you compute $\log_3(5)$?

18.[M]

(a) Using only the log key (and +, -, *, and /), compute $\log_2(6)$ and $\log_6(2)$.

(b) Compute the product of $\log_2(6)$ and $\log_6(2)$.

(c) Compute the product of $\log_7(11)$ and $\log_{11}(7)$.

(d) Make a conjecture about $\log_a(b) \cdot \log_b(a)$

(e) Show that the conjecture make in (d) is correct.

19.[C] Rarely is $\log_b(x+y)$ equal to $\log_b(x)$ plus $\log_b(y)$.

(a) Show that if $\log_b(x+y) = \log_b(x) + \log_b(y)$, then $y = x/(1-x)$.

(b) Give an example of x and y that satisfy the equation in (a).

The point of this Exercise is to show that while there is an identify for $\log_b(xy)$, there is no identity involving $\log_b(x+y)$.

20.[C] One way to compute b^4 is to start with b and keep multiplying by b three times, obtaining b^2, b^3, and, finally, b^4. But b^4 can be computed with only two multiplications. First compute b^2, then compute $b^2 \cdot b^2$. This raises the kind of question encountered when programming a computer. What is the fewest number of multiplications needed to compute b^n? Call that numer $m(n)$. For instance, $m(4) = 2$.

(a) Show that $m(2) = 1$, $m(3) = 2$, $m(5) = 3$, $m(6) = 3$, $m(7) = 3$, $m(8) = 3$, and $m(9) = 4$.

(b) Show that $m(n) \geq \log_2(n)$. Hint: Think of the final multiplication.

(c) Show that, when n is a power of 2, then $m(n) = \log_2(n)$. Hint: n is a power of 2 when $n = 2^k$, k a positive integer.

21.[C] Jane says to Sam, "I'm thinking of a whole number in the interval from 1 to 32. You have to find what iit is. I'll answer each question 'yes' or 'no'."

(a) What five questions, in order, should Sam ask to be sure he will guess the number?

(b) If, instead, the interval is from 1 to 50, how should Sam modify his questions to be guaranteed to guess the number in the fewest number of questions?

1.S Chapter Summary

This chapter reviewed precalculus material concerning functions. Calculus begins in the next chapter when we answer questions such as "What happens to $(2^x - 1)/x$ as x gets very small?". The answers are used in Chapter 3 to settle questions such as "How rapidly does 2^x change for a slight change in x?" That is where we meet the derivative of a function.

Section 1.1 introduced the terminology of functions: input (argument), output (value), domain, range, independent variable, dependent variable, piecewise-defined function, inverse of a function, graph of a function, decreasing, increasing, non-increasing, non-decreasing, positive, and monotonic.

Section 1.2 reviewed the key function x^k and its inverse $x^{1/k}$ (constant exponent, variable base), b^x (constant base, variable exponent) and its inverse $\log_b(x)$ and the six trigonometric functions and their inverses (for instance $\sin(x)$ and $\arcsin(x)$). All angles are measured in radians, unless otherwise stated.

Section 1.3 described five ways of getting new functions from two function f and g, namely $f+g$, $f-g$, fg, f/g, and the composition $f \circ g$.

Section 1.4 developed an explicit formula for a finite geometric sum with first term a and ratio r, $r \neq 1$: $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n))}{1-r}$.

Section 1.5 provided a quick review of the logarithm function to base b, b positive and $b \neq 1$, $\log_b$. This review emphasized the properties of logarithms.

EXERCISES for 1.S *Key:* R–routine, M–moderate, C–challenging

Exercises 1 to 10 concern logarithms, important functions in calculus and its applications. Remember that each property of a logarithm function is simply a translation of some property of an exponential function.

1.[R] Evaluate (a) $\log_3 \sqrt{3}$, (b) $\log_3(3^5)$, (c) $\log_3\left(\frac{1}{27}\right)$.

2.[R] If $\log_4 A = 2.1$, evaluate (a) $\log_4(A^2)$, (b) $\log_4(1/A)$, (c) $\log_4(16A)$.

3.[R] If $\log_3 5 = a$, what is $\log_5 3$?

4.[R] Find x if $5 \cdot 3^x \cdot 7^{2x} = 2$.

5.[R] Solve for x: (a) $2 \cdot 3^x = 7$, (b) $3^{5x} = 2^{7x}$, (c) $3 \cdot 5^x = 6^x$, (d) $10^{2x} 3^{2x} = 5$.

6.[R] Why do only positive numbers have logarithms? (In Chapter 12 negative numbers have logarithms also, provided with the aid of complex numbers.)

7.[R] Evaluate (a) $\log_2(2^{43})$, (b) $\log_2(32)$, and (c) $\log_2(1/4)$.

Exercises 8 to 10 concern the relation between logarithms to different bases.

8.[R] Suppose that you want to obtain $\log_2(17)$ in terms of $\log_3(17)$.

(a) Which would be larger $\log_2(17)$ or $\log_3(17)$?

(b) Show that $\log_2(17) = (\log_2(3))\log_3(1$
HINT: Take logarithms to the base 2 of both sic
of the equation $3^{\log_3(17)} = 17$.

9.[R]

(a) Calculate (by hand) $\log_a(b)$, $\log_b(a)$, and $\log_a($
$\log_b(a)$ when $a = 2$ and $b = 8$.

(b) Starting with $a^{\log_a(b)} = b$ and taking logarith
to the base b, show that $\log_a(b) \cdot \log_b(a) = 1$.

10.[R] You can use your calculator with a key for base-ten logarithms to compute logarithms to any base.

(a) Show why $\log_b(x) = \frac{\log_{10}(x)}{\log_{10}(b)}$.

(b) Compute $\log_2(3)$.

NOTE: When using the formula in (a) it is easy to forget whether you multiply or divide by $\log_{10}(b)$. As a memory device keep in mind that when b is "large," $\log_b(x)$ is "small," so you want to divide by $\log_{10}(b)$.

11.[R] If your scientific calculator lacks a key to display a decimal approximation to π, how could you use other keys to display it?

12.[R] Drawing pictures, find (a) $\tan(\arcsin(1/2))$, (b) $\tan(\arctan(-1/2))$, and (c) $\sin(\arctan(3))$.

13.[R] If f and g are decreasing functions, what (if anything) can be said about (a) $f + g$, (b) $f - g$, (c) f/g, (d) f^2, and (e) $-f$?

14.[R] What type of function is $f \circ g$ if (a) f and g are increasing, (b) f and g are decreasing, (c) f is increasing and g is decreasing? Explain.

15.[R] If f is increasing, what (if anything), can be said about $g = \mathsf{inv}(f)$?

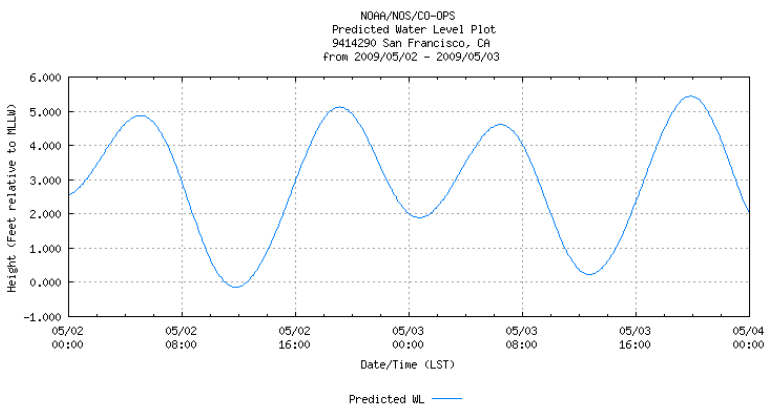


Figure 1.S.1: Source: `http://tidesandcurrents.noaa.gov/gmap3/`

16.[R] The predicted height of the tide at San Francisco for May 3, 2009 is shown in Figure 1.S.1.

(a) At what time(s) was the tide falling the fastest?

(b) At what time(s) was it rising the fastest?

(c) At what time(s) was it changing most slowly?

(d) How high was the highest tide? The lowest?

(e) At what rate was the tide going down at 2 p.m.? NOTE: Express this answer in feet per hour.

17.[R] Evaluate as simply as possible.

(a) $\log_3\left(3^{17.21}\right)$,

(b) $\log_5\left(5^{\sqrt{2}}/25^{\sqrt{3}}\right)$,

(c) $\log_2\left(4^{123}\right)$,

(d) $\log_2\left((4^5)^6\right)$,

(e) $\tan(\arctan(3))$.

18.[M] Give an example of (a) an increasing function f defined for positive x such that $f(f(x)) = x^9$ and (b) a decreasing function g such that $g(g(x)) = x^9$.

19.[M] Graph each of the following functions

(a) $\sin(x)$, x in $[0, 2\pi]$,

(b) $\sin(3x)$, x in $[0, \pi/2]$,

(c) $\sin(x - \pi)$, x in $[0, 2\pi]$,

(d) $\sin(3x - \pi/6)$, x in $[0, \pi/2]$.

(e) $y = \log_3(x)$, and

(f) $f(x) = \left(\frac{1}{2}\right)^x$.

20.[M] Imagine that the exponential key, x^y, on your calculator is broken. How would you compute $(2.73)^{3.09}$?

21.[M] The equation $y = x - \mathrm{e}\sin(x)$, known as **Kepler's equation**, is important in the study of the motion of planets. Here e is the eccentricity of an elliptical orbit, y is related to time, and x is related to an angle. For more information, visit `http://en.wikipedia.org/wiki/Keplerian_problem` or do a Google search for `Kepler equation`. NOTE: Kepler's equation, with $\mathrm{e} = 1$, reappears in Example 2 in Section 7.5 (see page 565).
The function $f(x) = x - \sin(x)$ is increasing for all numbers x. (See Exercise 26.)

(a) Graph f.

(b) Explain why, even though it cannot be found explicitly, you know the equation $y = x - \sin(x)$ can be solved for x as a function of y ($x = g(y)$).

(c) How are the graphs of $y = x - \sin(x)$ and $y = g(x)$ related?

22.[C] Copy and label each of the following in Figure 2.S.1(b).

(a) $y = x^2$,

(b) $y = x^3$,

(c) $y = 2^x$,

(d) $y = \log_2(x)$,

23.[C] The equation $\log_a(b) \cdot \log_b(a) = 1$ makes one wonder, "Is $\log_a(b) \cdot \log_b(c) \log_c(a) = 1$?" What is the answer? Either exhibit positive a, b, and c for which the equation does not hold or else prove it always hold.

24.[C] Find all numbers a and b such that $\log_a(b)$ equals $\log_b(a)$.

25.[C] A solar cooker can be made in the shape of part of a sphere. The one in Figure 1.S.2 spans only $\pi/3$ (60°) at the center $\mathcal{O}$. For simplicity, we take the radius to be 1.

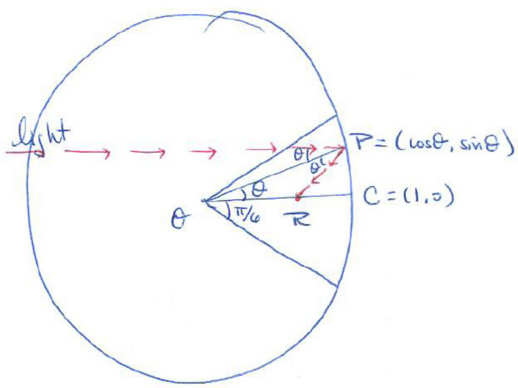


Figure 1.S.2:
Light parallel to $\mathcal{O}C$ strikes th
$(\cos(\theta), \sin(\theta))$ and is reflected to
radius $\mathcal{O}C$.

(a) There are two angles of meas
the top one equal to θ?

(b) Why is the bottom angle at

(c) Show that $\overline{\mathcal{O}R} = 1/(2\cos(\theta)$

(d) Show that the "heated part
length $(1/\sqrt{3}) - (1/2) \approx 0.0$
of the radius.

The Calculus is Everywhere sec
Chapter 3 describes a parabolic
flects all of the light to a single po

26.[C] Show that for $x < \pi/2$,
creasing function. Hint: Display
a unit circle, for two values of x,
also Exercise 21.

Calculus is Everywhere # 1
Graphs Tell It All

The graph of a function conveys a great deal of information quickly. Here are four examples, all based on numerical data.

The Hybrid Car

A friend of ours bought a hybrid car that runs on a fuel cell at low speeds and on gasoline at higher speeds and a combination of the two power supplies in between. He also purchased the gadget that exhibits "miles-per-gallon" at any instant. With the driver glancing at the speedometer and the passenger watching the gadget, we collected data on fuel consumption (miles-per-gallon) as a function of speed. Figure C.1.1 displays what we observed.

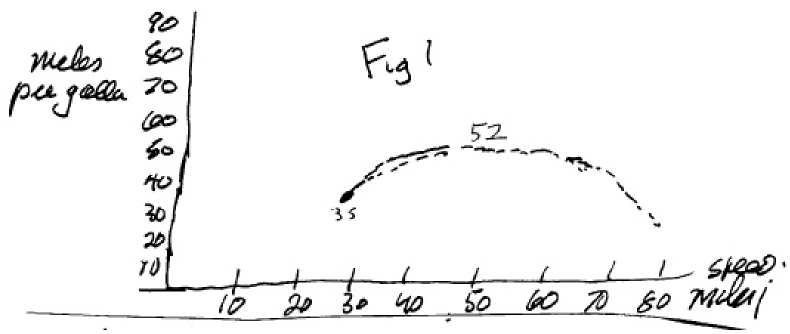


Figure C.1.1:

The straight-line part is misleading, for at low speeds no gasoline is used. So 100 plays the role of infinity. The "sweet spot," the speed that maximizes fuel efficiency (as determined by miles-per-gallon), is about 55 mph, while speeds in the range from 40 mph to 70 mph are almost as efficient. However, at 80 mph the car gets only about 30 mpg.

To avoid having to use 100 to represent infinity, we also graph gallons-per-mile, the reciprocal of miles-per-gallon, as shown in Figure C.1.2. In this graph the minimum occurs at 55 mph. And the straight line part of the graph on the speed axis (horizontal) records zero gallons per mile.

Life Insurance

The graphs in Figure C.1.3 compare the cost of a million-dollar life insurance policy for a non-smoker and for a smoker, for men at various ages. A glance at

By definition, a "non-smoker" has not used any tobacco product in the previous three years.

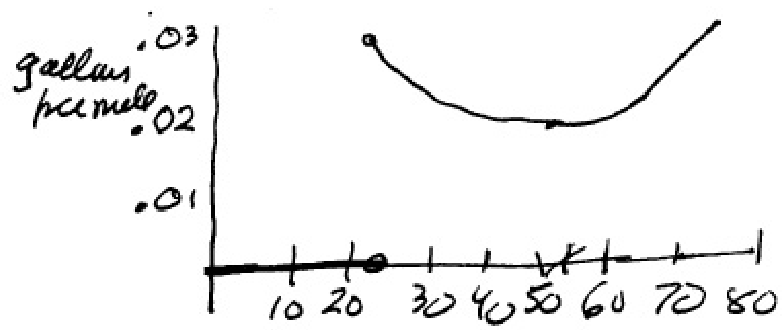


Figure C.1.2:

the graphs shows that at a given age the smoker pays about three times what a non-smoker pays. One can also see, for instance, that a 20-year-old smoker pays more than a 40-year-old non-smoker.

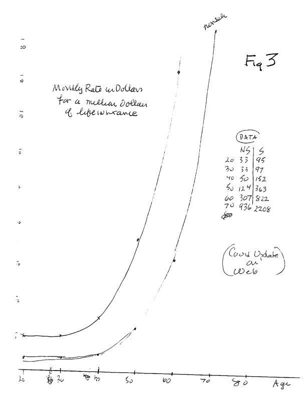


Figure C.1.3: Source: American General Life Insurance Company advertisement

Traffic and Accidents

Figure C.1.4 appears in S.K. Stein's, *Risk Factors of Sober and Drunk Drivers by Time of Day*, Alcohol, Drugs, and Driving 5 (1989), pp. 215–227. The vertical scale is described in the paper.

Glancing at the graph labeled "traffic" we see that there are peaks at the morning and afternoon rush hours, with minimum traffic around 3A.M.. However, the number of accidents is fairly high at that hour. "Risk" is measured by the quotient, "accidents divided by traffic." This reaches a peak at 1a.m.. The high risk cannot be explained by the darkness at that hour, for the risk rapidly decreases the rest of the night. It turns out that the risk has the same shape as the graph that records the number of drunk drivers.

It is a sobering thought that at any time of day a drunk's risk of being involved in an accident is on the order of one hundred times that of an alcohol-free driver at any time of day.

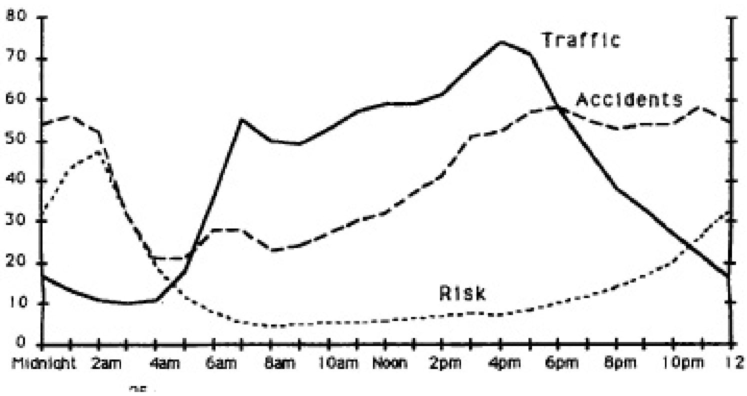


Figure C.1.4:

Petroleum

The three graphs in Figure C.1.5 show the rate of crude oil production in the United States, the rate at which it was imported, and their sum, the rate of consumption. They are expressed in millions of barrels per day, as a function of time. A barrel contains 42 gallons. (For a few years after the discovery of oil in Pennsylvania in 1859 oil was transported in barrels.)

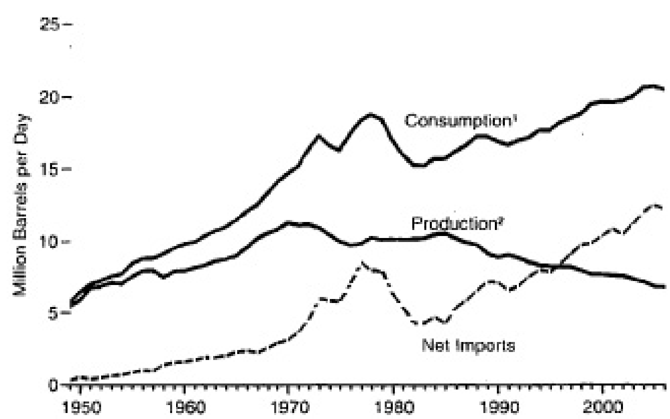


Figure C.1.5: Source: Energy Information Administration (Annual Energy Review, 2006)

The graphs convey a good deal of history and a warning. In 1950 the United States produced almost enough petroleum to meet its needs, but by

2006 it had to import most of the petroleum consumed. Moreover, domestic production peaked in 1970.

The imbalance between production and consumption raises serious questions, especially as exporting countries need more oil to fuel their own growing economies, and developing nations, such as India and China, place rapidly increasing demands on world production. Also, since the total amount of petroleum in the earth is finite, it will run out, and the Age of Oil will end. Geologists, having gone over the globe with a "fine-tooth comb," believe they have already found all the major oil deposits. No wonder that the development of alternative sources of energy has become a high priority.