# Piecewise-Defined Functions

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## *Overview*

A skydiver's height above ground is given by different formulae during the free-fall, the opening of the parachute, and the final descent. Mathematically, the height could be written as a single *piecewise-defined function*. The `piecewise` command for working with piecewise-defined functions is introduced in this lab. This will be helpful as you design a goblet.

## *Maple Essentials*

- New Maple commands introduced in this lab include:

| Command | Description |
|---|---|
| `convert` | converts an expression from one form to another form<br><br>To convert an expression into a piecewise-defined form use:<br>`convert(` $f$`, piecewise,` $x$ `);`<br><br>To convert a rational polynomial into its partial fraction decomposition use:<br>`convert(` $f$`, parfrac,` $x$ `);` |
| `piecewise` | define a piecewise-defined function<br><br>The general syntax to represent $\begin{cases} f_1, & cond_1 \\ f_2, & cond_2 \\ \vdots & \vdots \\ f_n, & cond_n \end{cases}$ is:<br>`piecewise(` $cond_1$ `,` $f_1$ `,` $cond_2$ `,` $f_2$ `, ...,` $cond_n$ `,` $f_n$ `);`<br>where each $cond_i$ is an inequality and each $f_i$ is an expression.<br><br>It is important to realize that Maple evaluates each $cond_i$ in order. If $cond_j$ is the first condition found to be `true`, the corresponding expression, $f_j$, is returned. |

## *Preparation*

Recall how to use the `VolumeOfRevolution` command to produce 3-D pictures of solids of revolution and definite integrals for their volume. Recall, from Calculus I, that a function, $f$, is *continuous at* $x = c$ exactly when $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$.

## *Assignment*

1. *If you have not done so already*, take a few minutes to complete the Post-Integration Applications Survey at the URL
   `http://distance-ed.math.tamu.edu/maple/dm152_integrationb/maple_quiz.htm`
   (Use your SAM login name as the login name for the quiz.)

2. Project 1 is due at the beginning of next week's lab. Remember to follow the Project Report Guidelines that are handed out today (and available on the lab homepage). Also, e-mail the Maple worksheet that creates your goblet to your lab TA.

3. For Mastery Quiz 5 you will be asked to write some expressions in the form of piecewise-defined functions.

## *Project 1: Goblet Design*

Your project is to design the most visually appealing goblet that meets the following criteria:

- the goblet will be molded using a symmetric mold, i.e., the goblet must be a solid of revolution
- the goblet must hold between 237 and 266 ml (8–9 oz) of your favorite liquid
- the height of the center of mass must be no more than 3 times the base radius i.e., the goblet must be reasonably stable,
- thickness of the glass must be at least $\frac{1}{4}$ cm at its thinnest point
- the goblet can be made with no more than 200 ml of glass
- the function for the upper curve of the region must be a piecewise-defined function with at least three "pieces", and at most one of the pieces can be a linear function. (Note that a constant function is a linear function.)

Your report should follow the guidelines set forth in the What is a Report Project? handout. In particular, your report should include the following:

- a complete description of the region to be revolved around the $x$-axis to construct the goblet
- a (2-D) plot of the region and a (3-D) plot of the goblet
- the amount of liquid that your goblet can hold and the amount of glass needed to make the goblet
- the minimum thickness of glass for your goblet
- the ratio of the height of the center of mass to the base radius

## *Activities*

1. Consider the function $G(x) = \left|x^2 - 4x\right|$. Use `diff` and `convert` to express the derivative of this function as a piecewise-defined function. Graph $y = G(x)$ and $y = G'(x)$ on the same set of axes. Are there any points where this function is not differentiable?
2. Plot the solid of revolution formed when the region bounded by the graph of $y = G(x)$, from Activity 1, the $x$-axis, $x = -1/2$, and $x = 3$ is rotated around the $x$-axis. Notice that this solid, when viewed with $\theta = 0$ and $\phi = 180$, is the shell of a goblet.
3. A martini glass is produced when the region bounded by the graphs of $y = F(x) = \begin{cases} 0.1 - 6x, & x < 0 \\ 0.1, & 0 \le x < 7 \\ 2x - 13.9, & x \ge 7 \end{cases}$, $y = G(x) = \begin{cases} 0, & x < 7 \\ 2x - 14, & x \ge 7 \end{cases}$, $x = -1/3$ and $x = 9$ is revolved around the $x$-axis.
   (a) Plot the region and the solid.
   (b) How much liquid will this goblet hold? How much glass is required to make this goblet?
   (c) What is the minimum thickness of glass in this goblet?
   (d) Let $R$ denote the radius of the base of the goblet. The height of the center of mass is located on the $x$-axis at $x = H$ where $H = \dfrac{\int_a^b (x-a)(f(x)^2 - g(x)^2)\,dx}{\int_a^b (f(x)^2 - g(x)^2)\,dx}$. Compute $R$, $H$, and $\frac{H}{R}$.
4. Consider the function $H(x) = \begin{cases} \sin(x), & x < \pi \\ h(x), & \pi \le x < 8 \\ 6 - \ln(x-5), & x \ge 8 \end{cases}$
   (a) When $h$ is a linear function, i.e., $h(x) = mx + b$, find values of the constants $m$ and $b$ that make $H$ a continuous function for all real numbers.
   (b) Plot the piecewise defined function found in (a) on the interval $-2\pi < x < 15$. Identify any points where this function is not differentiable.
   (c) When $h$ is a cubic function, i.e., $h(x) = ax^3 + bx^2 + cx + d$, find values of the constants $a$, $b$, $c$, and $d$ so that $H$ and its derivative $H'$ are both continuous for all real numbers.
   (d) Graph the functions found in parts (a) and (c) in the same set of axes.

## *Acknowledgement*

- This project is based on a project created in the Department of Mathematics at Kenyon College.