Inverse Functions and Their Derivatives

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Overview

The mathematical objective of this lab is for you to gain experience with inverse functions: identifying conditions when a function is invertible, finding an expression for the inverse function, and determining the derivative of an inverse function. From the Maple perspective, you will learn some new techniques to use Maple to manipulate and simplify expressions. In some cases it will be necessary for you to identify — and communicate to Maple — appropriate restrictions on the values of variables.

Maple Essentials

New Maple commands relevant to this material include:

Restriction	Description
`completesquare`	completes the square in an expression; in the `student` package
`isolate`	"solves" an equation for a variable or expression
`rhs`	returns the right hand side of an equation

Preparation

Review the material about inverse functions. In particular, be able to identify invertible functions using the Horizontal Line Test and the Monotonicity Test (Theorem 4.1.4, p. 246, in Anton). Know the steps involved in finding an inverse function and for finding the derivative of an inverse function.

Activities

Your TA will work through one or more of the following exercises.

1. The *hyperbolic cosine* and *hyperbolic sine* functions are defined as $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ and $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$. For each of these functions:
   (a) Find the largest domain on which the function is invertible.
   (b) Find an explicit formula for the inverse function. (Create a graph containing the function, its inverse, and $y = x$.)
   (c) Find the derivative of the function and the derivative of its inverse function.
2. For what values of the parameter k can the Monotonicity Test be used to conclude the function $f(x) = x^3 + kx^2 + x$ is invertible?
3. For what values of the parameter k can the Monotonicity Test be used to conclude the function $f(x) = x^3 + kx^2 - x$ is invertible?
4. Let $f(x) = \dfrac{ax + b}{cx + d}$.
   (a) Show that this function is invertible for all values of a, b, c, and d with $ad \neq bc$.
   (b) Find the general formula for $f^{-1}(x)$.
   (c) Find the general formula for $(f^{-1})'(x)$.

Additional Notes

- The derivatives of the hyperbolic functions, and their inverses, are generally written as

$$\begin{array}{rclrcl} \frac{d}{dx}\cosh(x) & = & \sinh(x) & \frac{d}{dx}\cosh^{-1}(x) & = & \dfrac{1}{\sqrt{x^2-1}} \\ \frac{d}{dx}\sinh(x) & = & \cosh(x) & \frac{d}{dx}\sinh^{-1}(x) & = & \dfrac{1}{\sqrt{x^2+1}} \end{array}$$

  While you might not obtain these expressions directly from Maple in Activity 1(c), you should be able to verify that your results are consistent with these formulae.

- The *Inverse* tutor can be used to plot a function and its inverse. This tutor can be started from the Maple 9.5 user interface under the Tools menu:

  Tools → Tutors → Calculus – Single Variable → Function Inverse ...

- The *DerivativeInverse* maplet provides practice finding the derivative of an inverse function. This maplet is available from Texas A & M University at the URL:

  `http://calclabs.math.tamu.edu/maple/maplets/DerivativeInverse.maplet`

- The `simplify` command was introduced earlier (during Week 7). Restrictions (assumptions) about the possible values for a variable can be imposed using an expanded version of the `simplify` command:

  `simplify(`*expression*`) assuming` *restrictions*`;`

  Some of the more common restrictions include:

Restriction	Description
`real`	assume *all* variables are real-valued
`a::real`	assume that a is a real-valued quantity
`a::positive`	assume that a is a positive quantity
`a::nonneg`	assume that a is a nonnegative quantity
`a>b`	assume that $a > b$
`a<>b`	assume that $a \neq b$

  Separate multiple restrictions with commas. For additional information, see the online help for `assuming`.

```
> q1 := exp( ln( x ) );                    # automatically simplifies to x
> q2 := ln( exp( x ) );                    # does not simplify automatically
> simplify( q2 ) assuming real;            # simplifies to x
```

Assignment

There is no assignment based on this lab. You should, however, spend some time to complete the remainder of the Activities.