Implicit Differentiation and Related Rates Problems

Objective   This lab presents two applications of the Chain Rule. It is designed to provide assistance with the technique of implicit differentiation, particularly as needed to answer related rates questions.

Background   Implicit differentiation requires the explicit identification of the independent variable. All other variables in the equation are either parameters (constants) or dependent variables (functions) of the independent variable.

Two different ways to perform implicit differentiation in Maple will be presented. One method mimics the steps one would take by hand to perform the computation (see Example 2). The second method is much easier, but involves the use of a new Maple command (see Example 2).

Related rates problems are word problems that involve rates of change, i.e., derivatives. The Chain Rule is usually used to find the critical relationship between quantities in the problem. In these problems Maple is best used to compute derivatives and to solve equations.

New Maple commands introduced in this lab are `implicitdiff` and `allvalues`. As the name suggests, the `implicitdiff` command returns a derivative computed using implicit differentiation. The simplest usage is `implicitdiff( eq, y, x )`; to determine $\frac{dy}{dx}$ where the equation *eq* implicitly defines y as a function of x. For example, the command

> `implicitdiff( x^2+y^2=1, y, x );`

returns $\frac{-x}{y}$ because $\frac{dy}{dx} = \frac{-x}{y}$ for all points on the unit circle (with $y \neq 0$). Maple uses the `RootOf` notation to represent all solutions of certain equations that have multiple solutions. The `allvalues` command is used to force Maple to display all values represented by a `RootOf`. For example,

> `    q := solve( { x+y=x*y, y-x=1 }, { x, y }`
> `);`
> `allvalues( q );`

REMINDER: *If an operation is easier to perform by hand — do so! Maple is a tool that is useful for some problems, but not everything. Do not feel that you need to use Maple just to use Maple.*

Discussion   Enter, and execute, the following Maple commands in a Maple worksheet.

Example 1: Implicit Differentiation with Implicit Dependence

```
> restart;                                   # clear Maple's memory
> eq1 := (x-a)^2 + (y-b)^2 = r^2;            # circle w/center (a,b), radius r
> DyDx1 := implicitdiff( eq1, y, x );        # assume y = y(x), compute dy/dx
> dy/dx = DyDx1;                             # display final result
> q1 := solve( {eq1,DyDx1=-1}, {x,y} );      # pts on circle w/m = -1
> allvalues( [q1] );                         # remove RootOf
```

Example 2: Implicit Differentiation with Explicit Dependence

```
> eq2 := eval( eq1, y=y(x) );                # circle w/explicit dependence
> Deq2 := diff( eq2, x );                     # diff both sides of eqn wrt x
> q2 := isolate( Deq2, diff(y(x),x) );        # solve for dy/dx
> DyDx2 := simplify( rhs( q2 ) );             # simplify result
> dy/dx = DyDX2;                              # display final result
> allvalues( [q3] );                          # remove RootOf
```

Example 3: Related Rates Problem

PROBLEM STATEMENT:
If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rates of change of the radius and volume when the diameter is 10 cm.

SOLUTION:
```
> Rvalue := r(t) = 5;                           # given value of radius
> Srate := diff( S(t), t ) = -1;                # given rate for surface area
> Seq := S(t) = 4*Pi*r(t)^2;                    # surface area (sphere, radius r)
> Veq := V(t) = 4/3*Pi*r(t)^3;                  # volume (sphere, radius r)
> dSeq := diff( Seq, t );                       # implicit differentiation
> dVeq := diff( Veq, t );                       # implicit differentiation
> q1 := eval( Veq, Rvalue );                    # volume when r = 5
> q2 := isolate( dSeq, diff( r(t), t ) );       # solve for dr/dt
> q3 := lhs(q2)                                 # dr/dt when dS/dt = -1, r = 5
>        = eval( rhs(q2), { Rvalue, Srate }
);
> eval( dVeq, { Rvalue, q3 } );                 # dV/dt at same instant
> eval( Veq, Rvalue );                          # dV/dt at same instant
```

Comments (right of code):
- `# given value of radius`
- `# given rate for surface area`
- `# surface area (sphere, radius r)`
- `# volume (sphere, radius r)`
- `# implicit differentiation`
- `# implicit differentiation`
- `# volume when r = 5`
- `# solve for` $\frac{dr}{dt}$
- `#` $\frac{dr}{dt}$ when $\frac{dS}{dt} = -1$, $r = 5$
- `#` $\frac{dV}{dt}$ at same instant
- `#` $\frac{dV}{dt}$ at same instant

Notes

(1) The `implicitplot` command is used to plot a curve when it is not possible, or not convenient, to write $y = f(x)$. The general syntax is `implicitplot( eqn, x=`*horizontalrange*`, y=`*verticalrange* `);`. If the resolution in this plot is poor, add the optional argument `grid=[`n_x`,`n_y`]` with reasonable values for n_x and n_y. The default grid is 25×25, for 625 points.

(2) REMINDER: When working related rates problems it is essential to compute all derivatives *before* values of dependent and independent variables are substituted into the calculation. Premature substitution will prevent the computation of the correct rate equation using implicit differentiation.

(3) The volume of a cone with radius r and height h is $V = \dfrac{\pi}{3}r^2 h$.

Questions

(1) Find $\dfrac{dy}{dx}$ when $\sqrt{y} + x^2 y = 5$.

(2) The curve $x^2 - xy + y^2 = 6$ is an ellipse centered at the origin. Find the equations of the tangent lines at the two points where the ellipse intersects the x-axis.

(3) A conical ice cream cone has a vertical axis, is 10 centimeters high, and has an opening with diameter 6 centimeters. The cone is completely filled with your favorite flavor of ice cream, but you do *not* eat the ice cream or the cone. Two hours later, a hole develops in the bottom of the cone and the melted ice cream drains at a rate of 2 cubic centimeters per second.

  (a) Find the relationship between the height h and radius r of the ice cream remaining in the cone.

  (b) Find the relationship between the rates of change of the height and radius of the ice cream remaining in the cone.

  (c) What is the rate of change of the height of (melted) ice cream left in the cone when the height is 5 centimeters?