Avoidable Words

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Strings of Symbols and Where They Might Come Up

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- Mathematical Logic.
- Computer Science.
- Automata and Formal Languages.
- Algebra.
- Topology.
- Number Theory.
- Dynamical Systems.

Patterns in Strings of Symbols

Most of the strings of symbols that arise in these displines display within themselves instances of patterns and other regularities.

It is the investigation of the occurrences of patterns in strings of symbols that is the topic of these lectures.

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Σ^+ denotes the set of all nonempty words over Σ.

Two Interesting Sequences of Words

The Prouhet-Thue-Morse Sequence

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The Prouhet-Thue-Morse Sequence

a

Two Interesting Sequences of Words

The Prouhet-Thue-Morse Sequence

a

ab

Two Interesting Sequences of Words

The Prouhet-Thue-Morse Sequence

a

ab

abba

Two Interesting Sequences of Words

The Prouhet-Thue-Morse Sequence

a

ab

abba

abbabaab

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a

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abba

abbabaab

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a

ab

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abbabaab

abbabaabbaababba

⋮

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a

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The Jacobson-Schutzenberger-Zimin Sequence

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a

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abbabaab

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Two Interesting Sequences of Words

The Prouhet-Thue-Morse Sequence

a

ab

abba

abbabaab

abbabaabbaababba

$\vdots$

The Jacobson-Schutzenberger-Zimin Sequence

a

aba

abacaba

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a

ab

abba

abbabaab

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⋮

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a

aba

abacaba

abacabadabacaba

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a
ab
abba
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a
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What is a Pattern?

A pattern is, in essence, a system of specifications that can be embodied and scaled in an assortment of ways.

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So patterns like xx are just words.

A word W is an **instance** or an **image** of a word U, provided W can be obtained from U by systematically substituting nonempty words for the letters of U.

Thus baabbaab is the instance of xx obtained by substituting the word baab for the letter x.

As another example

$$\mathrm{xyyxyxxy}$$

is the image of

$$\mathrm{abba}$$

via

$$a \mapsto xy \quad \text{and} \quad b \mapsto yx$$

Avoiding Patterns

We say a word V is a **subword** of a word W when W=XVY for some (possibly empty) words X and Y.

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Avoidable and Unavoidable Wrods

The word U is **avoidable on the alphabet** Σ provided infinitely many words in Σ^+ avoid U.

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The word U is **avoidable** if and only if U is k-avoidable for some natural number k.Words that fail to be avoidable are said to be **unavoidable**.

Are Any Words Avoidable? Are Any Words Unavoidable?

There better be some words of each kind!

Unavoidable	Word	Avoidable
	a	
	ab	
	aa	
	aba	
	aaa	
	abacaba	

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Indeed all the Zimin words, as well as any word encountered by a Zimin word, can be easily seen to be unavoidable.

The Adjacency Graph of a Word

The **adjacency graph** of a word W is a bipartite graph whose vertices are divided into two parts: a left copy of the alphabet and a right copy of the alphabet. The edges are determined by scanning the word. There is an edge joining the left copy of the letter x to the right copy of the letter y if and only if the word xy is a subword of the word W.

An example will serve to show how this works.

The Adjacency Graph of abacaba

As we slide a window of length 2 along abacabadabacaba we insert edges as shown.

abacabadabacaba

a	a
b	b
c	c
d	d

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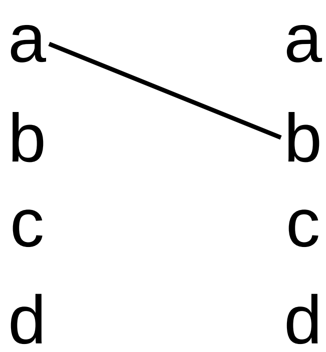

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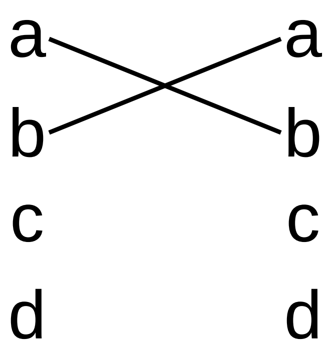

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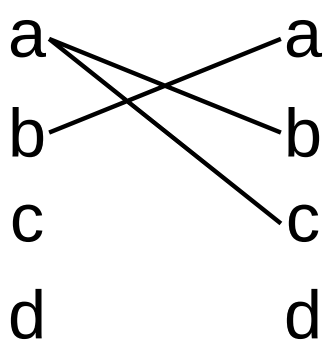

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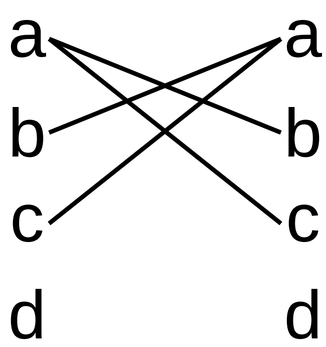

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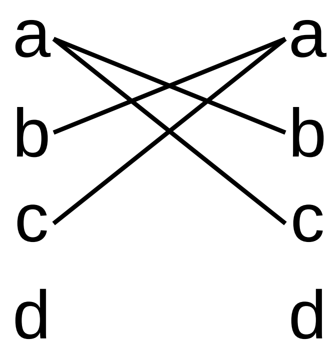

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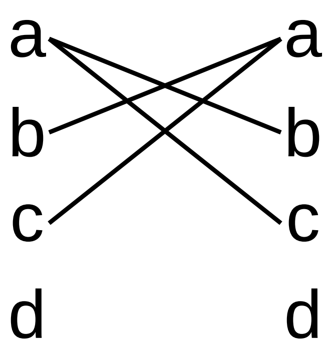

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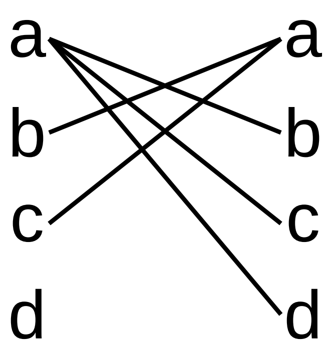

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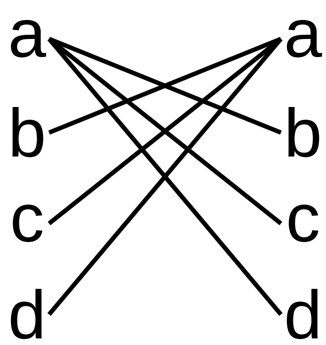
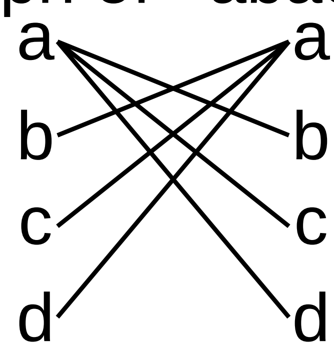


Free Sets

For a word W on an alphabet Σ we say that $F \subseteq \Sigma$ is **free** for W when no connected component of the adjacency graph of W contains both an element of the left copy of F and the right copy of F.

Since the adjacency graph of abacabadabacaba is



we see that the free sets are just all the subsets of these sets: $\{$a$\}$ and $\{$b,c,d$\}$.

Deletion of Letters from Words

Suppose W is a word and F is a set of letters. We can delete the letters in F from W by just erasing them. This leaves gaps, but by pushing the result together we get another word.

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We say a word W **reduces** to a word U if and only if U can be obtained from W by a sequence of one-step reductions.

A Sample Reduction

Now

- a is free for abacabadabacaba
- b is free for bcbdbcb
- c is free for cdc and finally
- d is free for d.

So the following reduction is available.

abacabadabacaba

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A Characterization of Unavoidability

Theorem (Bean, Ehrenfeucht, McNulty 1979; Zimin 1982). *Let W be a word. The following are equivalent:*

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- *W is unavoidable.*
- *W can be reduced to the empty word.*
- *The Zimin word on α letters encounters W, where α is the number of distinct letters in W.*

This theorem arose in the late 1970's in the work of Dwight Bean, Andrzej Ehrenfeucht, and George McNulty and independently in the work of I. A. Zimin. The last condition is due solely to Zimin.

Consequences of Unavoidability

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Call a word **long** if it has length at least 2^n where n is the number of distinct letters occurring in the word. We see that every long word is avoidable. The word xx is long even though it looks short. We have at least part of Thue's findings: xx is avoidable.

The Zimin word on n letters has length $2^n - 1$ and it is, as we noted, unavoidable. Indeed, it is as long as an unavoidable word on the n-letter alphabet can be.

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On the n-letter alphabet there are only finitely many unavoidable words. An industrious mathematician might try to describe them all. Indeed, one might go on to discover how many letters it takes to avoid the avoidable ones. After some preliminary work by U. Schmidt and P. Roth, J. Cassaigne managed to classify all words on 2 letters. As far as I know, nobody knows a complete description for words on 3 letters.

Today's Open Problem

What is the computational complexity of the set of unavoidable words?

A Short Side Trip: Squarefree Words

A word W is **squarefree** provided it avoids the pattern xx. Axel Thue proved in 1906 that there are infinitely many squarefree words on the 3 letter alphabet. Over the course of the ensuing 70 years Thue's work was rediscovered again and again. Since the mid-1970's many interesting things have been found out about squarefreeness. A look at these results can give some indication of what directions the study of avoidable words more generally might take.

How to Make Infinitely Many Squarefree Words

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then $h(U)$ *is squarefree whenever* $U \in \Sigma^+$ *is squarefree.*

Some Squarefree Homomorphisms

$$a \mapsto abcab$$

$$b \mapsto acabcb$$

$$c \mapsto acbcacb$$

Axel Thue, 1912

Some Squarefree Homomorphisms

$$a \mapsto abacb$$

$$b \mapsto abcbac$$

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Some Squarefree Homomorphisms

$$a \mapsto abcbacbcabcba$$

$$b \mapsto bcacbacabcacb$$

$$c \mapsto cabacbabcabac$$

Jonathan Leech, 1957

Some Squarefree Homomorphisms

$$
\begin{aligned}
a &\mapsto abcd \\
b &\mapsto abdc \\
c &\mapsto acbd \\
d &\mapsto acdb \\
e &\mapsto adbc
\end{aligned}
$$

Bean, Ehrenfeucht, McNulty, 1979

Some Squarefree Homomorphisms

$$
\begin{aligned}
a_0 &\mapsto dW_0eW_0 \\
a_1 &\mapsto dW_1eW_1 \\
a_2 &\mapsto dW_2eW_2 \\
&\vdots \qquad \vdots
\end{aligned}
$$

The words $W_0, W_1, W_2, \ldots$ is just the infinite list of squarefree words of $\{a, b, c\}$.

Bean, Ehrenfeucht, McNulty, 1979

Some Squarefree Homomorphisms

There is even a squarefree homomorphism from words on 4 letters to words on 3 letters. But you don't want to see it. The images of each of the three letters are words of length about 200.

Theorem (Bean, Ehrenfeucht, McNulty 1979). *Let Σ be the countably infinite alphabet and let Γ be the three-letter alphabet. There is a squarefree homomorphism from Σ^+ into Γ^+.*

Ordering Words

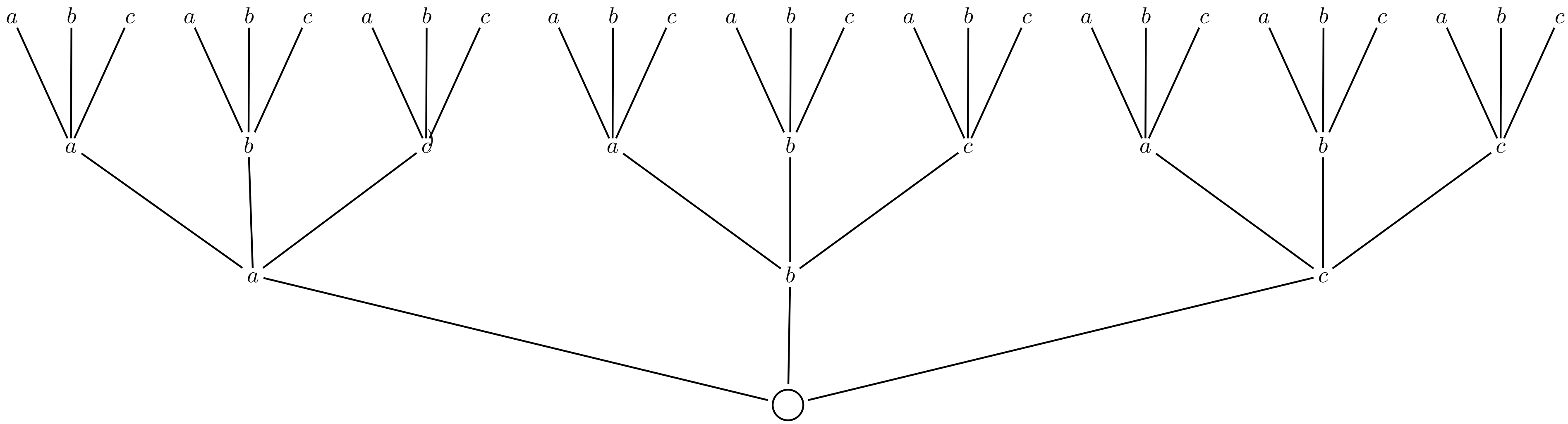
There are several interesting orders that can be imposed on sets of words. One is the ordering by initial segment. We say that the word W is an **initial segment** of the word U provided $U = WV$ for some, possibly empty, word U. Let us write $W \leq U$ when W is an initial segment of U.
It is easy to see that $\leq$ is reflexive, transitive and antisymmetric. That is $\leq$ is an order.
So for any alphabet Σ we see that $\langle \Sigma^*, \leq \rangle$ is an ordered set.

The Tree of Words

One way to display $\langle \Sigma^*, \leq \rangle$ is as a (rooted, ordered)tree. Suppose $\Sigma = \{a, b, c\}$ We impose the ordinary linear (alphabetical) ordering on Σ. Then we take the full ternary tree and label the nodes with the letters following a left to right pattern. The root we leave unlabelled.

The first few levels of this tree are displayed next.

The Bottom of the Tree



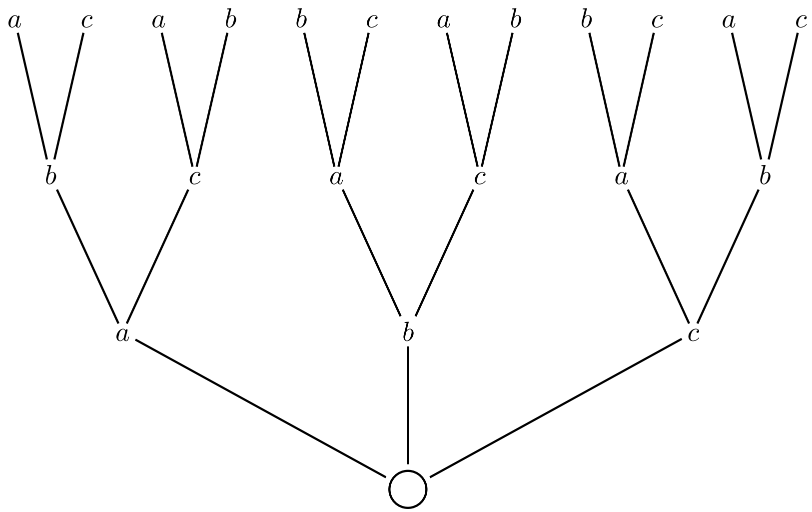
Walking Through the Tree

A walk starting at the root and passing up along the branches produces a word, letter by letter, as you pass over the branch points.

The tree has infinitely many levels. We could identify the infinite branches through the tree with the points in the unit interval (in this case expressed in ternary notation rather than decimal notation). This bring topology into the picture.

We could also agree to only include those walks along the branched which avoided some word W.

The Bottom of the Squarefree Tree



Finite Branches

Think of the last tree as a pruned version of the full ternary tree. Right now it looks like the full binary tree. But draw in a few more levels and you see that it is not.

For instance, the branch that reads $abacaba$ cannot be extended. It is a maximal squarefree word.

The number $g(n)$ of nodes at level n is the number of squarefree words of length n. The function g is the growth function for words avoiding xx.

We could do the same for every word W in place of xx.

Theorem (Bean, Ehrenfeucht, McNulty 1979). *Let W be a k^{th}-powerfree word on the finite alphabet Σ. W is a subword of some maximal k^{th}-powerfree word on Σ.*

The Growth of Squarefree Words

Theorem (Brandenburg 1983; Brinkhuis 1983). *On the three-letter alphabet, the growth rate of squarefree words has an exponential lower bound.*

In the Space of Infinite Words. . .

Theorem (Shelton and Soni 1980-83). *The space of squarefree ω-words on the three letter alphabet is perfect.*

Another Order

Let $W \triangleleft U$ mean that U ancounters W. This relation on words in reflexive and transitive. It fails however to be antisymmetric. But we can fix this.

Regard to words as **literally similar** provided there is a one-to-one map of the alphabet of one word onto the alphabet of the other that carries the first word to the second.

So cat and dog are literally similar while $read$ and $reed$ are not.

Modulo literal similarity $\triangleleft$ is an ordering on words.

The Ordered Set of Unavoidable Words

Let Σ be a finite alphabet. The set Γ of unavoidable words is Σ^* is finite. Up to literal similarity $\langle \Gamma, \lhd \rangle$ is a finite ordered set. It would be interesting to know something about its structure.

Today's Open Problem

For each natural number n find out how many unavoidable words there are on the n letter alphabet

How Many Letters Does It Take?

Let W be any word. We use $\alpha(\mathsf{W})$ to denote the number of distinct letters appearing in W.

If W is avoidable, then there is a smallest natural number $\mu(\mathsf{W})$ such that there is an infinite word on μ letters which avoids W.

We can extend this function μ to all words by putting $\mu(\mathsf{W}) = \infty$ when W is unavoidable.

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PROBLEM: What is the assymptotic behaviour of μ with respect to α?

The Word $\Omega_{m,z}$

The words mentioned in the title of this slide extend infinitely to the right. The parameters m and z are natural numbers with $m \geq 2$ and $z \geq 1$. The number of distinct letters in $\Omega_{m,z}$ is $m\lceil m^{1/z}\rceil$. I am only going to show you $\Omega_{2,1}$. It's alphabet has size 4. It proves convenient to use $\{0, 1, 2, 3\}$ as the alphabet.

The Word $\Omega_{m,z}$

Let φ be the endomorphism given by

$$
\begin{aligned}
0 &\mapsto 01\\
1 &\mapsto 21\\
2 &\mapsto 03\\
3 &\mapsto 23
\end{aligned}
$$

The Word $\Omega_{m,z}$

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Start from 0 and iterate φ:

$$0$$
$$01$$
$$0121$$
$$01210321$$
$$0121032101230321$$

Dean's Word: $\Omega_{2,1}$

$$0$$

$$01$$

$$0121$$

$$01210321$$

$$0121032101230321$$

$$\cdots$$

Dean's Word: $\Omega_{2,1}$

$$0$$

$$01$$

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$$\cdots$$

Notice that at each stage the previous word occurs as an initial segment. (This is a consequence of the fact that the image of 0 begins with 0.) Continuing in this way we construct the infinite word $\Omega_{2,1}$.

Three Theorems

Theorem (Baker, McNulty, Taylor 1989). *For integers* r, m, z *with* $r, z > 0$ *and* $m > (r+1)z$*, the word* $\Omega_{m,z}$ *avoids all avoidable words on* r *or fewer letters.*

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Theorem (I. Mel'nichuk 1988). $\mu < \alpha + 6$.

Walter's Word Δ

Let Δ be the word abwbcxcaybazac. It is a bit easier to understand Δ like this:

ab w bc x ca y ba z ac

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Walter's word Δ is not 3-avoidable.

Theorem (Baker, McNulty, Taylor 1989).

- *The growth rate of words on 4 letter which avoid Δ has a quadratic lower bound and a polynomial upper bound.*
- *On four letters the space of $\mathbb{Z}$-words avoiding Δ is perfect, and in fact a Cantor space.*

The Words of Ronald Clark, 2001

Δ is 4-avoidable but not 3-avoidable. That is $\mu(\Delta) = 4$. Let rho be the following word:

ab u ba w ac x bc y cda z dcd

Ronald Clark, in his Ph.D. dissertation written at UCLA in 2001 under the direction of Kirby Baker, proved that rho is 5-avoidable but not 4 avoidable.

Some Applications

Burnside's conjecture was that every finitely generated group of finite exponent is finite. This conjecture originated in 1905 and was not resolved (in the negative) until the work of Adjan and Novikov in 1968. The Adjan Novikov construction is genuinely elaborate, but it relies on the avoidability of xx and xxx.

Some Applications

Burris and Nelson (1971), and Jarda Jezek (1976) used the avoidability of xx to show that lattice of subvarieties of the variety of semigroups must have intervals of very high complexity. In fact Jezek produces an infinite set $\mathcal{F}$ of squarefree words on the three letter alphabet so that $\mathcal{F} - \{W\}$ avoids W for every $W \in \mathcal{F}$.

Some Applications

Mark Sapir (1987) used the full weight of avoidable words to give an alogithm for determining which finite semigroups have inherently nonfinitely based equational theories.

Some Applications

With the help of the theory of avoidability, Margolis and Sapir (1995) proved that no finite semigroup can be inherently nonfinitely Q-based.

Some Applications

A recent unpublished result of Ralph McKenzie is that given any function f on the natural numbers there is a finite semigroup $\mathbf{S}$ So that the free spectrum function of $\mathbf{S}$ dominates f.

Open Problems

PROBLEM 0: What is the computational complexity of the set of avoidable words?

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PROBLEM 2: Is μ a recursive function?

PROBLEM 3: Does μ achieve arbitrarily large values? Does it even get past 5?

PROBLEM 4: Let W be an avoidable word with $\mu(\text{W}) = m$. Can every W-free word on m letters be extended to a maximal W-free word on m letters?

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