# A Dozen Easy† Problems

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Department of Mathematics
University of South Carolina

American Mathematical Society
Special Session, St. Paul
10 April 2010

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Problems worthy
of attack
prove their worth
by hitting back.

—Piet Hein

These problems haven't tired of the fight! My reason for gathering them here is, of course, to prompt further work on them.

Problems about undecidability, computational complexity, and finite axiomatizability dominate this list—reflecting my own idiosyncratic fascination with the limits of what is mathematically describable.

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# Outline

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# Untried Problems

### The Finite Algebra Equational Completeness Problem

What is the computational complexity of determining whether a finite algebra of finite signature generates a minimal variety?

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### The Problem of the Likelihood of Even Factorability

Is $\frac{1}{2}$ the global asymptotic density of finite modular lattices that are evenly factorable?

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## A Concrete Combinatorial Finite Basis Problem

Is $\langle \mathbb{N}, +, \cdot, \binom{n}{k}, !, 0, 1 \rangle$ finitely based?

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Six Untried Problems

- The Finite Algebra Equational Completeness Problem
- The Problem of the Likelihood of Even Factorability
- A Concrete Combinatorial Finite Basis Problem
- The Finite Lattice Embeddability Problem
- The Free Lattice Nullstellen Problem
- Automatic Dualizability Problems

Six Untired Problems

- The Quackenbush Problem in Finite Signatures
- The Jónsson-Park Finite Basis Problem
- The Eilenberg-Schützenberger Finite Basis Problem
- The Modular Finite Height Unique Factorization Problem
- The Finite Congruence Lattice Problem
- The Lattice of Equational Theories Problem

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Is it decidable, given a finite lattice $\mathbf{L}$ and a finite algebra $\mathbf{A}$ of finite signature, whether $\mathbf{L}$ is embeddable into the congruence lattice of some algebra in the variety generated by $\mathbf{A}$?

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Prove a Nullstellensatz for free lattices.

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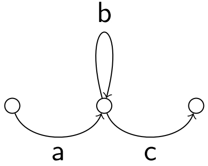
Which finite automatic algebras are dualizable? Is the automatic algebra drawn below dualizable?



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# The Finite Algebra Equational Completeness Problem

In 1956 Dana Scott published a brute force algorithm, relying on earlier work of Jan Kalicki, for determining whether a finite algebra of finite signature generates a minimal variety.

This algorithm, given a finite algebra $\mathbf{A}$, consists of first constructing the algebra free on 2 generators in the variety generated by $\mathbf{A}$ and then, using Kalicki's algorithm, checking that $\mathbf{A}$ belongs to the variety generated by each nontrivial quotient of the free algebra.

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# The Finite Algebra Equational Completeness Problem

In 1956 Dana Scott published a brute force algorithm, relying on earlier work of Jan Kalicki, for determining whether a finite algebra of finite signature generates a minimal variety.

On its face, this algorithm has at least doubly exponential time complexity (see the 2000 work of Bergman and Słutzki ). Perhaps the 1997 work of Kearnes and Szendrei points the way toward a more efficient algorithm.

# The Finite Algebra Equational Completeness Problem

Let $\mathbf{A}$ be a finite algebra which is equationally complete. Then for any algebra $\mathbf{B}$ of the same signature

$$\mathbf{B} \in \mathcal{HSP}\mathbf{A} \Leftrightarrow \mathbf{A} \times \mathbf{B} \text{ is equationally complete.}$$

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This gives us a Karp reduction (a many-one polynomial time reduction) between the Finite Algebra Membership Problem for **A** and the Equational Completeness Problem.

# The Finite Algebra Equational Completeness Problem

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# The Finite Algebra Equational Completeness Problem

In 1998, Zoltán Székely found a seven-element algebra with an NP-complete membership problem. By 2004, Marcin Kozik had constructed examples which are as difficult as possible: 2EXP-complete. Unfortunately, none of these examples appears to be equationally complete.

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On the other hand, in 1979 Don Pigozzi and in 1993 Ágnes Szendrei have provided examples of finite algebras which are equationally complete and not finitely based.

# The Finite Algebra Equational Completeness Problem

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# Untried Problems

### The Problem of the Likelihood of Even Factorability

Is $\frac{1}{2}$ the global asymptotic density of finite modular lattices that are evenly factorable?

# The Problem of the Likelihood of Even Factorability

A finite algebra $\mathbf{A}$ is *uniquely factorable* if and only if whenever

$$\mathbf{A} \cong \prod_{i \in I} \mathbf{B}_i \cong \prod_{j \in J} \mathbf{C}_j$$

where each $\mathbf{B}_i$ and each $\mathbf{C}_j$ is directly indecomposable, it follows that there is a bijection $\phi : I \to J$ so that $\mathbf{B}_i \cong \mathbf{C}_{\phi(i)}$ for all $i \in I$.

# The Problem of the Likelihood of Even Factorability

Let $\mathbf{A}$ be uniquely factorable. The algebra $\mathbf{A}$ is said to be *evenly factorable* provided $I$ has an even number of elements whenever

$$\mathbf{A} \cong \prod_{i \in I} \mathbf{B}_i \text{ where each } \mathbf{B}_i \text{ is directly indecomposable.}$$

# The Problem of the Likelihood of Even Factorability

Let $\mathcal{K}$ be any class of finite algebras with the unique factorization property. Let $\mathcal{K}(n)$ be the number of algebras in $\mathcal{K}$ with $n$ or fewer elements, counted up to ismorphism. Let $\mathcal{K}_E(n)$ be the number of algebras in $\mathcal{K}$ that are evenly factorable, that have $n$ or fewer elements, counted up to isomorphism.

# The Problem of the Likelihood of Even Factorability

The *global asymptotic density of even factorability* is

$$\lim_{n \to \infty} \frac{\mathcal{K}_E(n)}{\mathcal{K}(n)}.$$

It is not even clear that this limit exists.

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# The Problem of the Likelihood of Even Factorability

Ralph Freese has pointed to evidence that suggests that the random finite lattice is simple and hence directly indecomposable, making it oddly factorable. This leads him to conjecture that the asymptotic density of even factorability for $\mathcal{K}$ the class of all finite lattices is 0.

# The Problem of the Likelihood of Even Factorability

The global asymptotic density of even factorability for the class of finite Boolean lattices is $\frac{1}{2}$. The determination of the global asymptotic density of even factorability for the class of finite complemented modular lattices might be accessible. The situation when $\mathcal{K}$ is the class of finite distributive lattices, the class of finite groups, the class of finite Abelian groups, or the class of finite sets (finite algebras with the empty system of operations) seem to be open.

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# Untried Problems

### A Concrete Combinatorial Finite Basis Problem

Is $\langle \mathbb{N}, +, \cdot, \binom{n}{k}, !, 0, 1 \rangle$ finitely based?

# A Concrete Combinatorial Finite Basis Problem

A rich resource for combinatorial identities is the 1996 book $A = B$ by Petkovšek, Wilf, and Zeilberger. This book, which takes a computational perspective, provides powerful methods for establishing certain sorts of equations that hold in concrete combinatorial algebras like the one given above.

# A Concrete Combinatorial Finite Basis Problem

Here is one identity taken from page 34 of the book $A = B$:

$$\sum_{r,s}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+r}{r}\binom{n+s}{s}\binom{2n-r-s}{n} = \sum_{k}\binom{n}{k}^4.$$

Each of the sums here is finite. Of course, this identity involves subtraction and the operation $\sum$, which has no fixed rank. But only a bit of manipulation is needed to see this as an infinite schema of equations (one for each value of $n$) each holding in the concrete algebra of the problem formulated here.

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# The Finite Lattice Embeddability Problem

One might also ask several related problems:

- Is there a finite algebra $\mathbf{A}$ of finite signature such that the set of finite lattices embeddable into the congruence lattices of algebras in the variety generated by $\mathbf{A}$ is undecidable?
- Is there a locally finite variety $\mathcal{V}$ of finite signature such that the set of finite lattices embeddable into the lattices of congruences of algebras in $\mathcal{V}$ is undecidable?
- Is it decidable, given a finite lattice $\mathbf{L}$ and a finite algebra $\mathbf{A}$ of finite signature, whether $\mathbf{L}$ is isomorphic to the congruence lattice of some algebra in the variety generated by $\mathbf{A}$?

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# Untried Problems

### The Free Lattice Nullstellen Problem

Prove a Nullstellensatz for free lattices.

# The Free Lattice Nullstellen Problem

Let $\mathbf{L} = \mathbf{FL}(n)$ be the lattice generated by $n$ free generators. We take the elements of this free lattice to be the lattice terms (in canonical form) on the variables $x_0, x_1, \ldots, x_{n-1}$. A lattice equation in these variables is, in essence, just a pair of elements of $\mathbf{L}$.

# The Free Lattice Nullstellen Problem

Now consider an $n$-tuple $u = (u_0, \ldots, u_{n-1})$ of elements of $\mathbf{L}$ and an equation $s(x_0, \ldots x_{n-1}) \approx t(x_0, \ldots, x_{n-1})$. We say that $u$ is a *solution* of the equation provided

$$s^{\mathbf{L}}(u_0, \ldots, u_{n-1}) = t^{\mathbf{L}}(u_0, \ldots, u_{n-1}).$$

This condition is equivalent to asserting that $s(u_0, \ldots, u_{n-1}) \approx t(u_0, \ldots, u_{n-1})$ is an equation true in all lattices.

# The Free Lattice Nullstellen Problem

The binary relation "$u$ is a solution to $s \approx t$" sets up a Galois connection between the set $\mathbf{L} \times \mathbf{L}$ of equations and the set $\mathbf{L}^n$ of $n$-tuples.

# The Free Lattice Nullstellen Problem

The polarities of this Galois connection are

$$\Sigma^{\rightarrow} = \{u \in \mathbf{L}^n \mid u \text{ is a solution to } s \approx t \text{ for all } s \approx t \in \Sigma\}$$
$$S^{\leftarrow} = \{s \approx t \mid u \text{ is a solution to } s \approx t \text{ for all } u \in S\}$$

for all $\Sigma \subseteq \mathbf{L} \times \mathbf{L}$ and all $S \subseteq \mathbf{L}^n$.

# The Free Lattice Nullstellen Problem

${}^{\rightarrow\leftarrow}$ and ${}^{\leftarrow\rightarrow}$ are closure operators on $\mathbf{L} \times \mathbf{L}$ and $\mathbf{L}^n$ respectively. What this problem calls for is a characterization of these closure operators, particularly of the closure operator ${}^{\rightarrow\leftarrow}$.

# The Free Lattice Nullstellen Problem

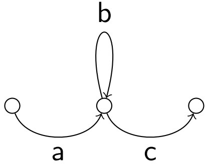
### The Problem Restated

Characterize $\theta^{\rightarrow\leftarrow}$ for all congruences $\theta$ of the free lattice on $n$ generators.

# Untried Problems

## Automatic Dualizability Problems

Which finite automatic algebras are dualizable? Is the automatic algebra drawn below dualizable?



# Automatic Dualizability Problems

An *automatic algebra* is an algebra **A** with only one basic operation, which is a two-place operation, that satisfies the following constraints.

1. The universe $A$ is the union of three pairwise disjoint sets $Q, \Sigma$, and $\{0\}$. $Q$ is called the set of states, $\Sigma$ is called the alphabet, and 0 is the default element.
2. $$uv \in \begin{cases} Q \cup \{0\} & \text{if } u \in Q \text{ and } v \in \Sigma \\ \{0\} & \text{otherwise,} \end{cases}$$
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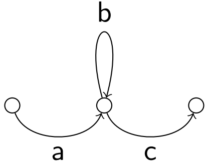

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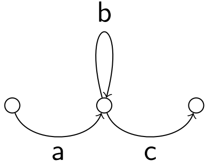


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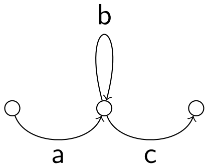


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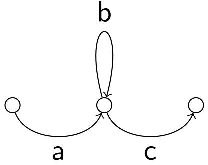


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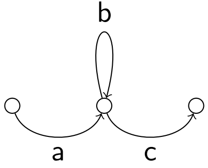


In the algebra drawn above, $Q$ consists of the three nodes represented as circles, $\Sigma = \{a, b, c\}$, the element 0 is not shown. If the nodes are named $q, r$ and $s$, reading left to right, then the operation of the algebra can be understood by following the arrows:

$$qa = r$$
$$rb = r$$
$$rc = s$$

with all other products resulting in the default element 0.

# Automatic Dualizability Problems

The essential problem framed here is to determine, in a computable manner, which automatic algebras are dualizable.

# Automatic Dualizability Problems

Recently John Boozer, in work so far unpublished, has undertaken a detailed study the finite basis properties of finite automatic algebras. Boozer's investigations show that the automatic algebra drawn above is not finitely based, but it also it just fails to be inherently nonfinitely based. This makes it an interesting candidate to investigate from the point of view of dualizability.

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