The equational compatibility problem for the real line

George F. McNulty

Department of Mathematics
University of South Carolina

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Signatures and Compatibility

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By a **signature** we mean a system of operation symbols each equipped with a natural number to specify its rank (i.e. its number of arguments). Thus a signature is a function that assigns ranks to operation symbols.

Let Δ be the set of equations listed below, which axiomatize the theory of commutative rings with unit.

$$\begin{array}{cc}
x+(y+z) \approx (x+y)+z & x\cdot(y\cdot z) \approx (x\cdot y)\cdot z \\
x+y \approx y+x & x\cdot y \approx y\cdot x \\
x+(-\mathbf{1}+\mathbf{1}) \approx x & x\cdot \mathbf{1} \approx x \\
x+(-x) \approx -\mathbf{1}+\mathbf{1} & x\cdot(y+z) \approx x\cdot y + x\cdot z
\end{array}$$

Evidently, Δ is compatible with the real line, while the set $\Delta \cup \{x\cdot y \approx \mathbf{0}\}$ is not compatible with the real line.

The Equational Compatibility Problem for ρ

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This problem is **algorithmically unsolvable** if no such algorithm exists.

Walter Taylor's Breakthrough

Let τ be the signature of rings with unit expanded by three unary operation symbols and let τ' be the expansion of τ by the countably infinite list $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \dots$ of constant symbols.

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Taylor's Compatibility Theorem

The equational compatibility problem for the signature τ' is algorithmically unsolvable.

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Is there an algorithm for determining of any equation γ of signature ρ whether γ is compatible with the real line?

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Refining Taylor's Theorem

Let σ be the signature of rings with unit expanded by two unary operation symbols.

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The Single Equational Compatibility Theorem

The single equational compatibility problem for the signature σ is algorithmically unsolvable.

Two Algebras

Walter Taylor's work centered on the topological algebra

$$\mathbf{R}_{\text{Taylor}} = \langle \mathbb{R}, +, \cdot, -, 0, 1, \sin^*, \cos^*, \lambda \rangle$$

where $\sin^*(\chi) := \sin(\frac{\pi}{2}\chi)$ and $\cos^*(\chi) := \cos(\frac{\pi}{2}\chi)$ for all real numbers χ, and where λ is a certain function which satisfies $(\lambda(\chi))^2 = \cos^*(\sin^*(\chi))$ for all real numbers χ.

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We replace this algebra by

$$\mathbf{R} = \langle \mathbb{R}, +, \cdot, -, 1, \sin^*, |\ | \rangle$$

dropping 0 and $\cos^*$ in the interests of parsimony, since they can be defined in terms of the remaining operations.

The Threefold Method: Finite Determination

We say a topological algebra $\mathbf{T}$ with universe $\mathbb{R}$ is **finitely determined** provided there is a finite set Σ of equations which is compatible with $\mathbf{T}$ and up to isomorphism (simultaneously algebraic and topological) $\mathbf{T}$ is the only topological algebra with universe $\mathbb{R}$ which is a topological model of Σ. We also say that $\mathbf{T}$ is determined by Σ.

Let Δ_0 be the set consisting of the following equations

$$\begin{aligned}
\sin^*(x+y) - \sin^*(x-y) &\approx 2\sin^* y \sin^*(x+1)\\
\sin^* 1 &\approx 1\\
(\sin^*(\sin^*(x)+1)) &\approx |\sin^*(\sin^*(x)+1)|\\
|x \cdot x| &\approx x \cdot x\\
|x| &\approx |-x|
\end{aligned}$$

Let Δ_{ra} be the set consisting of the following equations

$$\begin{gathered}\sin^*(x+y) - \sin^*(x-y) \approx 2\sin^* y \sin^*(x+1)\\ \sin^* 1 \approx 1\\ \sin^*(1+\sin^*(x)) \approx (\boldsymbol{\lambda}(x))^2\\ \boldsymbol{\lambda}(x+4) \approx \boldsymbol{\lambda}(x)\\ \boldsymbol{\lambda}(0) \approx -1\\ \boldsymbol{\lambda}(2) \approx 1\end{gathered}$$

Determination Theorems

Taylor's Finite Determination Theorem

The topological algebra $\mathbf{R}_{\text{Taylor}}$ is determined $\Delta \cup \Delta_{\text{ra}}$.

Determination Theorems

Alternate Finite Determination Theorem

The topological algebra $\mathbf{R}$ is determined by $\Delta \cup \Delta_0$.

Undecidability Theorems

Taylor's Undecidability Theorem

There is no algorithm which will determine of any equation in signature τ whether it has a solution in $\mathbf{R}_{\mathrm{Taylor}}$.

Undecidability Theorems

Alternate Undecidability Theorem

There is no algorithm which will determine of any equation in one variable in signature σ whether it is true in $\mathbf{R}$.

The One Equation Collapse

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This theorem was discovered around 1966 by George Grätzer and Ralph McKenzie and, independently, by Alfred Tarski

For any equation ε of signature τ let ε' be the equation obtained by replacing each occurance of the variable x_i by the constant symbol c_i for each natural number i. The following are equivalent

1. ε is solvable in $\mathbf{R}_{\text{taylor}}$.
2. $\Delta \cup \Delta_{\text{ra}} \cup \{\varepsilon'\}$ is compatible with the real line.

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Proving Taylor's Equational Compatibility Theorem

For contradiction, suppose there were an algorithm to solve the equational compatibility problem for τ'. Here is how to decide whether an equation ε of signature σ is solvalbe in $\mathbf{R}_{\text{Taylor}}$.

1. Put $\Sigma = \Delta \cup \Delta_{\text{ra}} \cup \{\varepsilon'\}$.
2. Determine whether Σ is compatible with the real line.

For any equation ε of signature σ the following are equivalent

1. ε is true in $\mathbf{R}$.
2. $\Delta \cup \Delta_0 \cup \{\varepsilon\}$ is compatible with the real line.
3. γ_Σ is compatible with the real line, where $\Sigma = \Delta \cup \Delta_0 \cup \{\varepsilon\}$.

For any equation ε of signature σ the following are equivalent

1. ε is true in $\mathbf{R}$.
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For any equation ε of signature σ the following are equivalent

1. ε is true in $\mathbf{R}$.
2. $\Delta \cup \Delta_0 \cup \{\varepsilon\}$ is compatible with the real line.
3. γ_Σ is compatible with the real line, where $\Sigma = \Delta \cup \Delta_0 \cup \{\varepsilon\}$.

Proving the Single Equational Compatibility Theorem

For contradiction, suppose there were an algorithm to solve the single equational compatibility problem for σ. Here is how to decide whether an equation ε of signature σ is true in $\mathbf{R}$.

1. Put $\Sigma = \Delta \cup \Delta_0 \cup \{\varepsilon\}$.
2. Determine whether γ_Σ is compatible with the real line.

What It Takes to Determine $\mathbf{R}_{\text{Taylor}}$ and $\mathbf{R}$

In essence, Walter Taylor established both Determination Theorems. As Taylor observes, much of what is needed belongs to the theory of functional equations and traces back to d'Alembert in the mid-1700's and to Cauchy's 1821 treatise. Here are some particularly helpful things to know:

1. The Intermediate Value Theorem.
2. The Mean Value Theorem.
3. The Integrability of Continuous Functions.
4. Riemann Sums
5. The solutions to second-order ordinary differential equations with constant coefficients.
6. Some elementary trigonometry.

Establishing Undecidability

The Negative Solution of Hilbert's 10^{th} Problem

There is no algorithm for determining of an equation (in 9 unknowns) whether it is solvable in the ring of integers.

Martin Davis, Hilary Putnam, and Julia Robinson laid out a method to prove this in 1961 and Yuri Matiyasevich provided the last piece needed to complete their method in 1970. Matiyasevich's 1993 book provides a careful account of the resolution of this problem and some of its consequences. Since the integers can be easily defined in $\mathbf{R}_{\text{Taylor}}$ with the help of $\sin^*$, Taylor's Undecidability Theorem follows readily.

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In 1968, Daniel Richardson originated the central ideas behind Alternate Decidability Theorem. Given the negative solution to Hilbert's 10^{th} Problem, Richardson's ideas rely only on familiar calculus—this time the Mean Value Theorem for Several Variables. Richardson's ideas have to be adapted to the algebra $\mathbf{R}$.

Achieving the One Equation Collapse

The methods of Grätzer and McKenzie and of Tarski are somewhat different. However, either of these methods depends solely on clever manipulations of equations and should be accessible to anyone with an abstract algebra course in their background.

More Smoothness

More smoothness than continuity can be demanded. A set Γ of equations (or a single equation γ) is $\mathcal{C}^n$-**compatible** with the real line provided there is an algebra $\mathbf{T}$ with universe $\mathbb{R}$ whose operations are all in $\mathcal{C}^n$ so that Γ (or γ) is true in $\mathbf{T}$. We could even demand real analytic operations to get **real analytic compatibility**.

More than algorithmic unsolvability

Let A and B be two sets of elements which can be inputs to algorithms. We say that A and B are **algorithmically inseparable** provided for every set C of allowable inputs such that C includes A and is disjoint from B there is no algorithm to settle the membership problem of C.

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Taylor's Inseparability Theorem

The collection of finite sets of equations of signature τ' which are not compatible with the real line is algorithmically inseparable from the collect of finite sets of equations of signature τ' which are real analytic compatible with the real line.

Two More Inseparability Theorems

The $\mathcal{C}^n$ Inseparability Theorem

For each natural number n, the set of equations in signature σ which are incompatible with the real line is algorithmically inseparable from the set of equations which are $\mathcal{C}^n$-compatible with the real line.

Two More Inseparability Theorems

The Real Analytic Inseparability Theorem

The set of equations in signature σ' which are incompatible with the real line is algorithmically inseparable from the set of equations which are real analytic compatible with the real line.

Two More Inseparability Theorems

The Real Analytic Inseparability Theorem

The set of equations in signature σ' which are incompatible with the real line is algorithmically inseparable from the set of equations which are real analytic compatible with the real line.

Here σ' be the expansion of σ by one additional constant symbol $\mathbf{c}$.