# Finite Axiomatizability
# for
# The Class of Subdirectly Irreducible Groups
# Generated by a Finite Group

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This equivalence is a classical result of Garrett Birkhoff.

# Subdirect Representations

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The homomorphic images $h_i(\mathbf{A})$ are called the **factors** of the representation.

A subdirect representation is **trivial** if one of its homomorphisms is an isomorphism.

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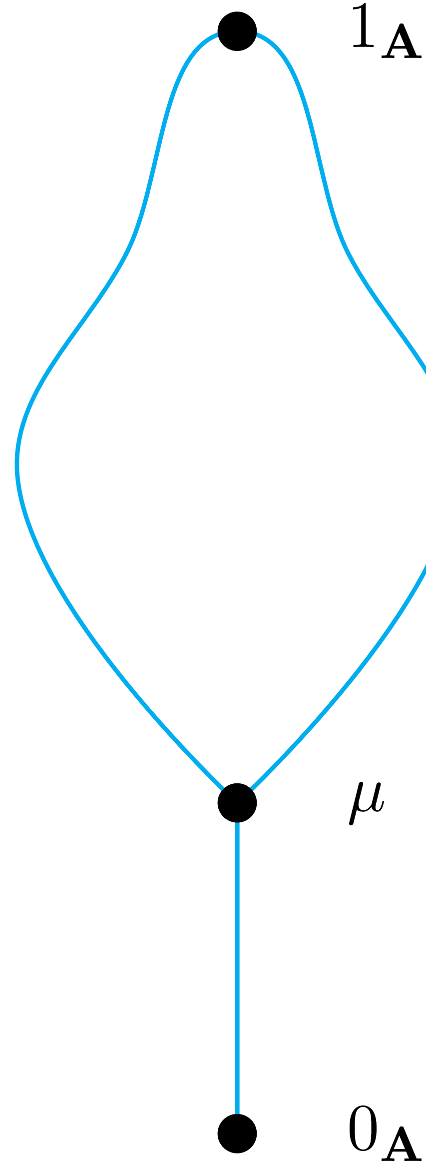
The subdirectly irreducible algebras are exactly those which have a **critical pair**, i.e. a pair $(a, b)$ of distinct elements such that any homomorphism which is not one-to-one must assign to $a$ and $b$ the same value.

# The Congruence Lattice of a Subdirectly Irreducible Algebra



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For any class $\mathcal{K}$ of algebras we use $\mathcal{K}_{\text{si}}$ to denote the class of subdirectly irreducible algebras in $\mathcal{K}$.

# Our Main Theorem

If $\mathcal{V}$ is the variety generated by a finite group, then $\mathcal{V}_{\text{si}}$ is a finitely axiomatizable elementary class.

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McKenzie (1970)

According to a classical result of Bjarni Jónsson (1967), there is a finite upper bound on the cardinalities of members of $\mathcal{V}_{\mathrm{si}}$, when $\mathcal{V}$ is generated by a finite lattice (even expanded by finitely many operation symbols).

On the other hand, if $\mathcal{V}$ is generated by a finite group and contains a nilpotent group which is not Abelian, then $\mathcal{V}$ contains arbitrarily large infinite subdirectly irreducible groups.

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# The Proof of Our Main Theorem

According to the Theorem of Oates and Powell, there is an elementary sentence $\Sigma$ which axiomatizes $\mathcal{V}$. Let $\Phi$ be the sentence guaranteed by the Real Main Theorem. Evidently, $\Sigma \curlywedge \Phi$ axiomatizes $\mathcal{V}_{\mathsf{si}}$.

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Let $\mathbf{H}$ be a group and let $a, b \in H$. We use $\mathrm{Cg}^{\mathbf{H}}(a, b)$ to denote the smallest congruence of $\mathbf{H}$ which identifies $a$ and $b$.

By a **unary polynomial of $\mathbf{H}$** we mean a function $t^{\mathbf{H}}(x, a_0, a_1, \ldots)$ where $t(x, y_0, y_1, \ldots)$ is a term and $a_0, a_1, \cdots \in H$. The **complexity** of this polynomial is the length of the shortest term $t(x, y_0, y_1, \ldots)$ that can be used to represent it.

When $(c, d) \in \mathrm{Cg}^{\mathbf{H}}(a, b)$ because $c = q(a)$ and $d = q(b)$ we say that the polynomial $q(x)$ **witnesses** this membership constraint.

**Theorem 1.** *Let* $\mathbf{H}$ *be a group and* $a, b \in H$*. Then*

$$\mathrm{Cg}^{\mathbf{H}}(a,b) = \{(q(a), q(b)) : \textit{for some unary polynomial } q(x)\}.$$

# Ladders in Groups

Let $\mathbf{H}$ be any group. A system $B = \langle B_0, B_1, \ldots, B_{m-1} \rangle$ of finite subsets of $H$ is said to be a **ladder** provided

1. $1 \in B_i$ and $B_i$ has at least two elements, for each $i < m$,
2. $B_i \cap B_j = \{1\}$ for all $i, j < m$ with $i \neq j$,
3. $(z, 1) \in \mathrm{Cg}^{\mathbf{H}}(x, y)$ for all $x \in B_i, y \in B_j$ and $z \in B_k$ with $1 \neq x \neq y$ and for all $i, j, k < m$ with $j, k \leq i$.

The congruences of the form $\mathrm{Cg}^{\mathbf{H}}(x, y)$ where $x \in B_i$ and $y \in B_j$ with $j \leq i < m$ and $x \neq y$ are called the **principal conguences** of the ladder.

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Evidently, $\varphi_0 \subseteq \varphi_1 \subseteq \cdots \subseteq \varphi_{m-1}$ is chain of congruences. By the **length** of the ladder $B$ we mean the length of this chain of congruences.

**Theorem 2.** *If the variety* $\mathcal{V}$ *is generated by the finite group* $\mathbf{G}$ *and* $\mathbf{H} \in \mathcal{V}$*, then* $|G|$ *is an upper bound on the length of any ladder of* $\mathbf{H}$*. Moreover,* $|G|$ *also bounds the length of any chain of completely join irreducible elements of* $\mathrm{Con}\,\mathbf{H}$*.*

Let $\mathbf{H}$ be any group and $a \in H$. The **ladder height** of $a$ is the least upper bound on the lengths of ladders all of whose principal congruences are included in $\mathrm{Cg}^{\mathbf{H}}(a, 1)$.

We will denote the ladder height of $a$ in $\mathbf{H}$ by $\rho^{\mathbf{H}}(a)$.

Notice that the ladder height of $1$ is $0$.

**Theorem 3.** *Let* $\mathbf{G}$ *be a finite group and let* $\mathcal{V}$ *be the variety generated by* $\mathbf{G}$*. There is a finite bound* $n$ *such that for any* $\mathbf{A} \in \mathcal{V}$ *and any* $a \in A$ *with* $\rho^{\mathbf{A}}(a) > 1$*, there is* $b \in A$ *with* $b \neq 1$ *and* $\rho^{\mathbf{A}}(b) < \rho^{\mathbf{A}}(a)$ *such that there is a polynomial of complexity no more than* $n$ *which witnesses* $(b, 1) \in \mathrm{Cg}^{\mathbf{A}}(a, 1)$*.*

**Theorem 4.** *Let* $\mathbf{H}$ *be a group and suppose* $a, b, c, d \in H$ *so that* $(c, d) \in \mathrm{Cg}^{\mathbf{H}}(a, b)$*. Then there is a polynomial of complexity no more than* $4|1/\mathrm{Cg}^{\mathbf{H}}(a, b)| + 1$ *that witnesses this membership constraint.*