The computational complexity of deciding whether a finite algebra generates a minimal variety.

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Outline

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Computational Problems About Finite Algebras

The Minimal Variety Problem

- An Upper Bound
- A Lower Bound

A Conjecture

The Minimal Variety Problem

Input: *A finite algebra* $\mathbf{A}$ *of finite signature.*

Problem: *Decide if the variety generated by* $\mathbf{A}$ *is minimal.*

What is the computational complexity of this problem?

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In 1955, Dana Scott observed that there is a brute force algorithm to decide this problem.

The Tarski's Finite Basis Problem

Input: *A finite algebra* **A** *of finite signature.*

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What is the computational complexity of this problem?

Way too hard! McKenzie showed in 1993 that there is no algorithm for deciding this

The Finite Algebra Membership Problem For A Finite Algebra $\mathbf{B}$ of Finite Signature

Input: *A finite algebra* $\mathbf{A}$ *of the signature of* $\mathbf{B}$.

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In 1953 Jan Kalicki observed that there is a brute force algorithm for solving this problem.

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In 1998, Zoltan Székely devised a seven-element algebra **B** for which this problem is NP-complete.

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In 2000, Cliff Bergman and Giora Słutzki found Kalicki's algorithm is in 2EXPTIME.

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In 2009, Marcin Kozik devised a finite algebra $\mathbf{B}$ for which this problem in 2EXPTIME-complete

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Problem: *Decide if the variety generated by* **A** *is congruence distributive.*

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According to folklore (but probably Bjarni Jónsson is the folk mentioned), there is a brute force algorithm to decide this.

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In 2009, Ralph Freese and Matthew Valeriote proved that this problem, as well as several similar problems, is EXPTIME-complete.

The Affine Complete Variety Problem

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In 2002, Kalle Kaarli and Alden Pixley gave a not quite brute force algorithm to decide this problem.

The Affine Complete Variety Problem

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What is the computational complexity of this problem?

It should be a homework problem for Ralph Freese and Matthew Valeriote to show that this problem is actually EXPTIME-complete.

The Minimal Variety Problem

Input: *A finite algebra* $\mathbf{A}$ *of finite signature.*

Problem: *Decide if the variety generated by* $\mathbf{A}$ *is minimal.*

What is the computational complexity of this problem?

Outline

Dana Scott's Brute Force Algorithm

Let **A** be a nontrivial finite algebra of finite signature. To decide whether $\mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{A}$ is a minimal variety

Step I Make a list $\mathbf{B}_0, \mathbf{B}_1, \ldots,$ up to isomorphism, of all the 2-generated algebras in $\mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{A}$.

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Let $\mathbf{A}$ be a nontrivial finite algebra of finite signature. To decide whether $\mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{A}$ is a minimal variety

- Step I: Construct a minimal nontrivial subalgebra $\mathbf{S}$ of $\mathbf{A}$.
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- Step III: Determine if $\mathbf{A} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{S}$. If not, punt.
- Step IV: Determine if every strictly simple algebra in $\mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{S}$ is isomorphic to $\mathbf{S}$. If so, then $\mathbf{A}$ generates a minimal variety. If not, punt.

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How hard can that be?

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Don't laugh!

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Steps I and II are easy.

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Theorem

The Minimal Variety Problem can be settled in 2EXPTIME.

Keith and Ágnes Offer Another Approach to Step IV

A unary term e in the signature of $\mathbf{A}$ will be called a *minimal idempotent* of $\mathbf{A}$ if the term operation $e^{\mathbf{A}}$ is idempotent, not constant, and the image $e^{\mathbf{A}}(A)$ of $e^{\mathbf{A}}$ is minimal with respect to inclusion among all sets of the form $f(A)$ for nonconstant, unary, idempotent term operations f of $\mathbf{A}$.

A Kearnes-Szendrei Theorem

Let $\mathbf{S}$ be a strictly simple algebra and e a minimal idempotent of $\mathbf{S}$. The following conditions are equivalent.

(1) $\mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{S}$ is minimal.

(2) (a) $\mathbf{S}$ is nonabelian or has a trivial subalgebra, and
    (b) For some $n > 1$, there exist binary terms f_i and unary terms g_i and h_i for all i with $0 \leq i \leq n$ such that the following equations hold in $\mathbf{S}$:

$$\begin{aligned} x &\approx f_0(x, eg_0(x)), \\ f_i(x, eh_i(x)) &\approx f_{i+1}(x, eg_{i+1}(x)) \text{ for all } i \text{ with } 0 \leq i < n, \\ f_n(x, eh_n(x)) &\approx e(x). \end{aligned}$$

This theorem of Kearnes and Szendrei offers another strategy for the Minimal Variety Problem. A stumbling block is that to verify the Mal'cev-like condition one must consider very many very long sequences of equations. That might well be a task in 2EXPTIME as well.

Outline

Theorem
The Minimal Variety Problem is NP-hard.

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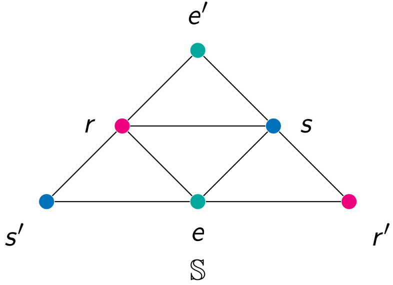
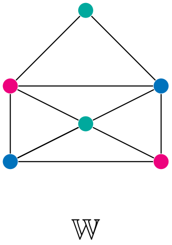
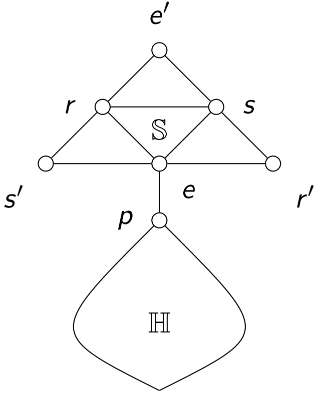
The proof reduces the minimal variety problem to the 3-colorability problem for finite connected graphs.





The Graphs $\mathbb{S}$ and $\mathbb{W}$



The Graph $\mathbb{S}_{\mathbb{H}}$

The Algebra $\mathbf{S}^\circ$

The universe of $\mathbf{S}^\circ$ is $S^\circ = S \cup \{0\}$, where 0 is not in $\mathbb{S}$.

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$$0, c_e, c_r, c_s, c_{e'}, c_{r'}, \text{ and } c_{s'}$$

to name each element of S.

The Algebra $\mathbf{S}^\circ$

The universe of $\mathbf{S}^\circ$ is $S^\circ = S \cup \{0\}$, where 0 is not in $\mathbb{S}$.

The signature of $\mathbf{S}^\circ$ has 8 binary operation symbols:

$$\cdot, \wedge, Q_e, Q_r, Q_s, Q_{e'}, Q_{r'}, \text{ and } Q_{s'}$$

to name the Shallon graph algebra operation, a height 1 meet operation, and the Pigozzi operations.

The Operations of $\mathbf{S}^\circ$

The Shallon operation:

$$u \cdot v = \begin{cases} u & \text{if there is an edge joining } u \text{ and } v \\ 0 & \text{Otherwise} \end{cases}$$

for all $u, v \in S^\circ$.

The Operations of $\mathbf{S}^\circ$

The height 1 meet:

$$u \wedge v = \begin{cases} u & \text{if } u = v \\ 0 & \text{Otherwise} \end{cases}$$

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The Operations of $\mathbf{S}^\circ$

The Pigozzi operation Q_e:

$$Q_e(u, v) = \begin{cases} v & \text{if } e = u \\ 0 & \text{Otherwise} \end{cases}$$

for all $u, v \in S^\circ$.

The Algebra $\mathbf{S}^\circ_{\mathbb{H}}$

This algebra has the same signature as $\mathbf{S}^\circ$. Its universe is $S_{\mathbb{H}} \cup \{0\}$ and its operations are defined just as those for $\mathbf{S}^\circ$. In particular, there are only 7 constant symbols and they still name the elements of S°. Notice that $\mathbf{S}^\circ$ is a subalgebra of $\mathbf{S}^\circ_{\mathbb{H}}$.

Plan of the Proof

We will prove that for any finite connected graph $\mathbb{H}$

$$\mathbb{H} \text{ is 3-colorable}$$

$$\text{if and only if}$$

$$\mathbf{S}^{\circ}_{\mathbb{H}} \text{ generates a minimal variety.}$$

Plan of the Proof

We do this in three stages:

1. $\mathbf{S}^\circ$ generates a minimal variety.
2. $\mathbf{S}^\circ_{\mathbb{H}} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{S}^\circ$ if and only if there is a natural number t and an embedding $\varphi : \mathbb{S}_{\mathbb{H}} \to \mathbb{S}^t$ with the property that $\varphi(a) = \langle a, \dots, a\rangle$ for each $a \in S$.
3. There is a natural number t and an embedding $\varphi : \mathbb{S}_{\mathbb{H}} \to \mathbb{S}^t$ with the property that $\varphi(a) = \langle a, \dots, a\rangle$ for each $a \in S$ if and only if $\mathbb{H}$ is 3-colorable.

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Then

$$\begin{aligned} \mathbf{S}^\circ_{\mathbb{H}} \text{ generates a minimal variety } &\Leftrightarrow \mathbf{S}^\circ_{\mathbb{H}} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{S}^\circ \\ &\Leftrightarrow \mathbb{H} \text{ is 3-colorable.} \end{aligned}$$

Step 1: Listen to Don Pigozzi

For this step show

$\mathbf{S}^\circ$ generates a minimal variety.

The idea is to show that $\mathbf{S}^\circ$ can be embedded into every nontrivial algebra $\mathbf{B} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{S}^\circ$ via the map that sends each element of S° to the element of $\mathbf{B}$ named by the corresponding constant symbol. The only real issue is to show that this map is one-to-one. It is the Pigozzi operations that save the day.

Step 2: Listen to Zoltan Székely Invoke Ralph McKenzie

For this step show

$\mathbf{S}^\circ_{\mathbb{H}} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{S}^\circ$ if and only if there is a natural number t and an embedding $\varphi : \mathbb{S}_{\mathbb{H}} \to \mathbb{S}^t$ with the property that $\varphi(a) = \langle a, \ldots, a \rangle$ for each $a \in S$.

For the right-to-left direction consider the subalgebra of $(S^\circ)^t$ generated by the image of $S_{\mathbb{H}}$. A glance at the operations reveals that this subalgebra consists of the elements of the image, all of which are proper t-tuples, as well as some improper t-tuples. The equivalence relation that lumps together the improper elements and isolates the proper elements is a congruence. The quotient algebra is isomorphic to $\mathbf{S}^\circ_{\mathbb{H}}$.

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For the left-to-right direction observe that $\mathbf{S}^\circ_{\mathbb{H}}$ is subdirectly irreducible, since $\mathbb{S}_{\mathbb{H}}$ is connected. Ralph McKenzie showed us how to pick a natural number t, a subalgebra $\mathbf{B}$ of $(\mathbf{S}^\circ)^t$, a congruence $\theta \in \operatorname{Con} \mathbf{B}$ and a proper element $p \in B$ so that

(a) $\mathbf{S}^\circ_{\mathbb{H}} \cong \mathbf{B}/\theta$

(b) For all $u, v \in B$ we have

$$u \;\theta\; v \text{ if and only if } \mu(u) = p \Leftrightarrow \mu(v) = p \text{ for all translations } \mu.$$

Take $\mathbf{B}$ as small as possible.

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$\mathbf{S}^\circ_{\mathbb{H}} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{S}^\circ$ if and only if there is a natural number t and an embedding $\varphi : \mathbb{S}_{\mathbb{H}} \to \mathbb{S}^t$ with the property that $\varphi(a) = \langle a, \ldots, a\rangle$ for each $a \in S$.

For the left-to-right direction:
Since translations of improper elements must be improper, we see that θ puts all the improper elements into the same congruence class that we will call the *zero-block*. McKenzie also tells us that θ isolates p. Using the Pigozzi operations we can show that none of the t-tuples like $\langle e, \ldots, e\rangle$ belong to the zero-block.

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Let U be the complement of the zero-block and let $\mathbf{B}'$ be the subalgebra of $\mathbf{B}$ generated by U and let θ' be the restriction of θ to B'. A glance at the operations reveals that B' consists of the elements of U together with certain improper tuples. But this means $\mathbf{B}/\theta \cong \mathbf{B}'/\theta'$. So by the minimality of $\mathbf{B}$ we see that $\mathbf{B} = \mathbf{B}'$.

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This means that $\mathbb{S}_{\mathbb{H}}$ is isomorphic to the subgraph of $\mathbb{S}^t$ induced by the proper elements of B via an isomorphism φ with the property that $\varphi(a) = \langle a, \dots, a\rangle$ for each $a \in S$.

Step 3: Listen to Zoltan Székely (but recall William Wheeler)

For this step show

There is a natural number t and an embedding $\varphi : \mathbb{S}_{\mathbb{H}} \to \mathbb{S}^t$ with the property that $\varphi(a) = \langle a, \ldots, a \rangle$ for each $a \in S$ if and only if $\mathbb{H}$ is 3-colorable.

For the left-to-right direction, the map $\varepsilon \circ \pi \circ \varphi$ turns out to be a 3-coloring of $\mathbb{S}_{\mathbb{H}}$, where π can be any of the projection functions, and ε is the function erasing primes (e.g., $\varepsilon(e') = e = \varepsilon(e)$).

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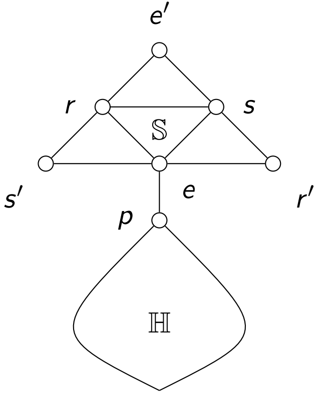

The Graph $\mathbb{S}_{\mathbb{H}}$

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The Constraints on the Array

(a) No two rows are exactly alike.

(b) The entries of the row associated with p are drawn from $\{r, r', s, s'\}$ and all four of these values occur as entries in that row.

(c) For each $k < t$ the k^{th} column of the array is a 3-coloring of $\mathbb{S}_{\mathbb{H}}$ once the primes are erased. (Well, . . .)

(d) For each vertex q of $\mathbb{H}$ other than p each of the values r', s', and e' occur among the entries of the row associated with q.

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How to handle the last constraint

Suppose that q and q' are vertices of $\mathbb{H}$ that are not adjacent. Pick a 3-coloring, using r, s, and e, of $\mathbb{H}$ that assigns r to the vertex p. Place this coloring in the column under construction.

$$\begin{array}{ccc} p & \to & r \\ \vdots & & \vdots \\ q & \to & s \\ \vdots & & \vdots \\ q' & \to & e \end{array}$$

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In $\mathbb{S}$ the vertices r, s, and e are pairwise adjacent, so this column as it stands would not disrupt the adjacency of the images of q and q'.

How to handle the last constraint

Modify the column by putting primes on the entries associated with q and q'. In $\mathbb{S}$, the no primed vertex is adjacent to a primed vertex, so this modified coloring entails that the images of q and q' will not be adjacent.

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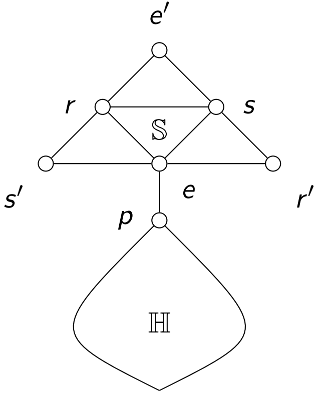
Have other, needed, adjacency been disrupted? No. Suppose q was assigned the color s. Then the vertices adjacent to q must have been assigned colors from $\{r, e\}$. But in $\mathbb{S}$ the vertex s' is adjacent to both the vertex r and the vertex e.



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Conjecture

The Minimal Variety Problem is 2EXPTIME complete.