Finitely based varieties and quasivarieties—or Ralph, tell us all that you know

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I just got here

Outline

Finitely Based Varieties

- A sample of finite basis theorems in the traditional manner
- Theorems that belong to general algebra: Finite basis theorems for varieties
- The Approach of Bjarni Jónsson, modified
- Dilworth and Mal'cev tell us how to generate principal congruences

Finitely Based Quasivarieties

- First things first
- Can it all be done again, for quasivarieties?
- Extending Ross Willard's Finite Basis Theorem to quasivarieties

Problems

Finally. . .

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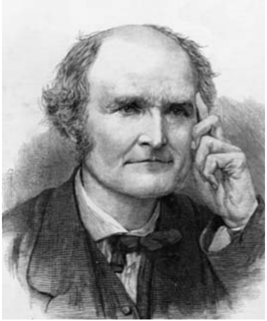
Finite basis theorems in the 19th century tradition



Arthur Cayley has a theorem, 1854

The class of all groups is finitely based.

Finite basis theorems in the 19$^{\text{th}}$ century tradition

That is not Martin Powell!

The Oates and Powell Finite Basis Theorem, 1965

Varieties generated by finite groups are finitely based.

Finite basis theorems in the 19$^{\text{th}}$ century tradition

Robert Kruse

Some L'vov

The Kruse, L'vov Finite Basis Theorem, 1973

Varieties generated by finite rings are finitely based.

Finite basis theorems in the 19th century tradition

One of McKenzie's Finite Basis Theorems, 1970

Varieties generated by finite lattices, with finitely many additional operations, are finitely based.

Outline

Garrett Birkhoff gets us started

Birkhoff's Finite Basis Theorem, 1935

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is locally finite,

then $\mathcal{V}^{(n)}$ is finitely based, for every natural number n.

Roger Lyndon picks up the ball

Lyndon's Finite Basis Theorem, 1951

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is generated by an algebra of cardinality 2,

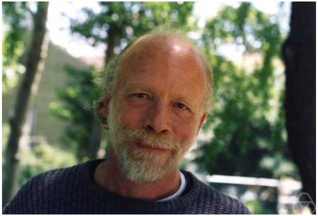
then $\mathcal{V}$ is finitely based.

Alfred Tarski asks a question

Tarski's Finite Basis Problem, 1950's

Does there exist an algorithm, which upon input of a finite algebra $\mathbf{A}$ of finite signature will determine whether the variety generated by $\mathbf{A}$ is finitely based?

Ralph McKenzie has an answer



McKenzie Resolution of Tarski's Finite Basis Problem, 1995

There is no algorithm, which, upon input of a finite algebra $\mathbf{A}$ of finite signature, will determine whether the variety generated by $\mathbf{A}$ is finitely based.

Kirby Baker takes a big step

Baker's Finite Basis Theorem, 1977

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence distributive, and
- $\mathcal{V}$ is finitely generated,

then $\mathcal{V}$ is finitely based.

Ralph McKenzie takes another

McKenzie's Finite Basis Theorem, 1987

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence modular,
- $\mathcal{V}$ is finitely generated, and
- $\mathcal{V}$ is residually small,

then $\mathcal{V}$ is finitely based.

Ralph McKenzie takes another

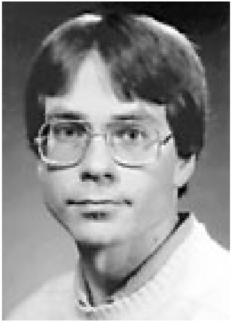
McKenzie's Finite Basis Theorem, 1987

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence modular,
- $\mathcal{V}$ has a finite residual bound,

then $\mathcal{V}$ is finitely based.

Ross Willard takes another direction



Willard's Finite Basis Theorem, 2000

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence meet-semidistributive, and
- $\mathcal{V}$ has a finite residual bound,

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 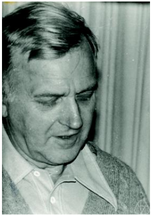

Jónsson's Finite Basis Theorem, 1979

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence distributive, and
- $\mathcal{V}_{\mathsf{fsi}}$ is finitely axiomatizable,

then $\mathcal{V}$ is finitely based.

Finitely Subdirectly Irreducible Algebras

An algebra is *finitely subdirectly irreducible* provided the intersection of any two nontrivial congruences of the algebra is itself nontrivial.

For any class $\mathcal{W}$ of algebras we use $\mathcal{W}_{\mathsf{fsi}}$ to denote the class of finitely subdirectly irreducible algebras belonging to $\mathcal{W}$.

Every subdirectly irreducible algebra is finitely subdirectly irreducible, but the converse fails often. Here is a useful consequence:

Fact

Varieties of the same signature that have the same finitely subdirectly irreducible algebras coincide.

A Working Lemma

Lemma

Let $\mathcal{V}$ and $\mathcal{W}$ be a classes of algebras of the same signature with $\mathcal{V} \subseteq \mathcal{W}$ such that

(a) $\mathcal{V}$ is a variety,

(b) $\mathcal{W}$ is finitely axiomatizable, and

(c) Both $\mathcal{V}_{\mathsf{fsi}}$ and $\mathcal{W}_{\mathsf{fsi}}$ are finitely axiomatizable.

Then $\mathcal{V}$ is finitely based.

A Proof

Let $\mathcal{W} = \operatorname{Mod} \sigma$, $\mathcal{W}_{\mathsf{fsi}} = \operatorname{Mod} \theta$, and $\mathcal{V}_{\mathsf{fsi}} = \operatorname{Mod} \varphi$. Then

$$\mathcal{V} \models \sigma \mathbin{\wedge\!\!\!\wedge} (\theta \to \varphi).$$

Let Σ be a finite set of equations true in $\mathcal{V}$ so that

$$\Sigma \vdash \sigma \mathbin{\wedge\!\!\!\wedge} (\theta \to \varphi).$$

Let $\mathcal{V}' = \operatorname{Mod} \Sigma$. Then $\mathcal{V}'$ is a finitely based variety with

$$\mathcal{V} \subseteq \mathcal{V}' \subseteq \mathcal{W}.$$

But it is easy to see that $\mathcal{V}_{\mathsf{fsi}} = \mathcal{V}'_{\mathsf{fsi}}$. This entails that $\mathcal{V} = \mathcal{V}'$. So $\mathcal{V}$ is finitely based.

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When is $\mathcal{W}_{\mathsf{fsi}}$ finitely axiomatizable?

Evidently, $\mathbf{A}$ is finitely subdirectly irreducible provided that the intersection

$$\mathrm{Cg}^{\mathbf{A}}(a, b) \cap \mathrm{Cg}^{\mathbf{A}}(c, d)$$

of any two nontrivial *principal* congruences is itself nontrivial.

Were there a formula $\kappa(x, y, z, w)$ is that

$$\mathbf{A} \models \kappa(p, q, a, b) \text{ if and only if } (p, q) \in \mathrm{Cg}^{\mathbf{A}}(a, b)$$

then one could write down another formula $\pi(x, y, z, w)$ so that

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Suppose $\kappa(x, y, , z, x)$ defined principal congruences for all the algebras in $\mathcal{V}$. If the signature of $\mathcal{V}$ is finite, one can write down a sentence σ that asserts that $\kappa(x, y, z, w)$ defines principal congruences. So σ will be true in $\mathcal{V}$.

Even so, we can only conclude that $\mathcal{V}_{\mathsf{fsi}}$ is finitely axiomatizable relative to $\mathcal{V}$.

However, if be take $\mathcal{W} = \mathrm{Mod}\,\sigma$, then $\mathcal{W}_{\mathsf{fsi}}$ will be finitely axiomatizable.

So all the stipulations in our lemma are fulfilled, apart from the stipulation that $\mathcal{V}_{\mathsf{fsi}}$ be finitely axiomatizable.

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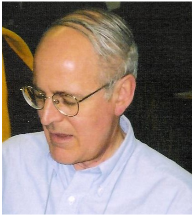
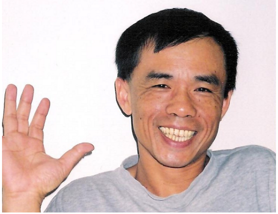
A Theorem of McKenzie, 1978

Let $\mathcal{V}$ be a variety of finite signature.
If

- $\mathcal{V}$ has definable principal congruences, and
- $\mathcal{V}_{\mathrm{fsi}}$ is finitely axiomatizable,

then $\mathcal{V}$ is finitely based.





A Theorem of Baker and Wang, 2002

Let $\mathcal{V}$ be a variety of finite signature. If

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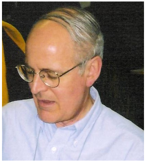

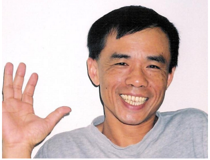

A Theorem of Baker, McNulty, and Wang, 2004

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence meet-semidistributive,
- $\mathcal{V}$ is locally finite,
- $\mathcal{V}$ has finitely bound on critical depth, and
- $\mathcal{V}_{\mathsf{fsi}}$ is finitely axiomatizable,

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Translations

Definition

Let $\mathbf{A}$ be an algebra. A function $\lambda : A \to A$ is a

Basic translation provided λ arises from a basic operation of $\mathbf{A}$ by evaluating all but one of its arguments with elements of A.

k-translation provided λ can be realized as the composition of a sequence of k or fewer basic translations.

Translation provided λ is a k-translation for some natural number k.

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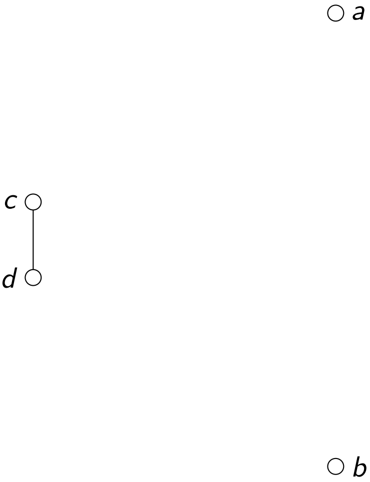
Observe that the identity function is the only 0-translation.

Dilworth and Mal'cev

According to Dilworth and Mal'cev



$\langle a, b \rangle \in \mathrm{Cg}^{\mathbf{A}}(c, d)$

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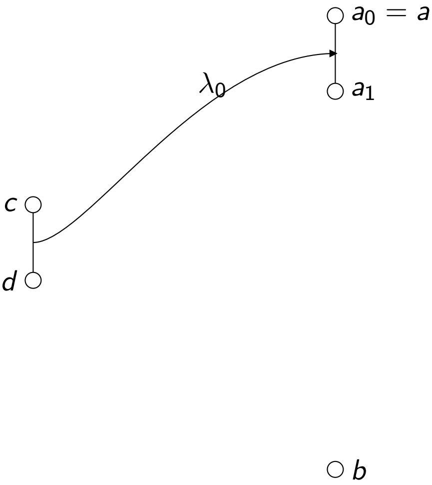

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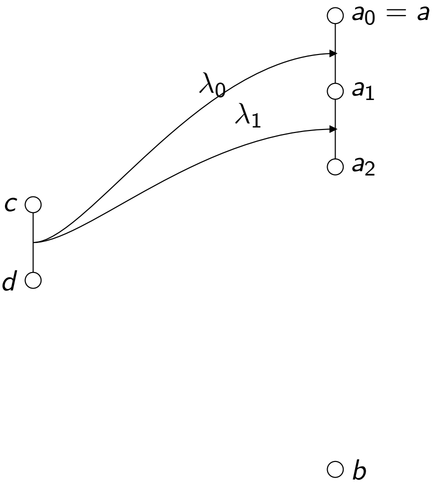

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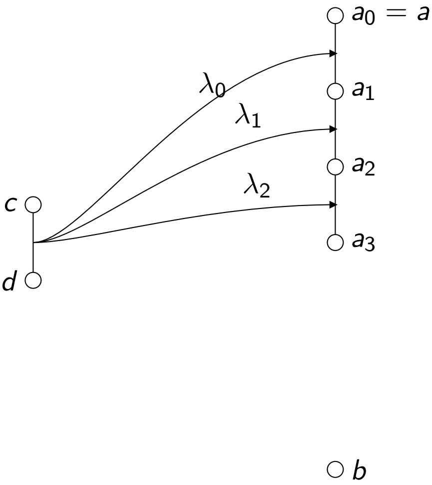

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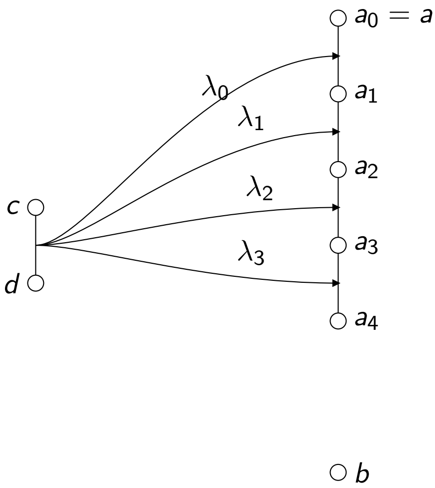

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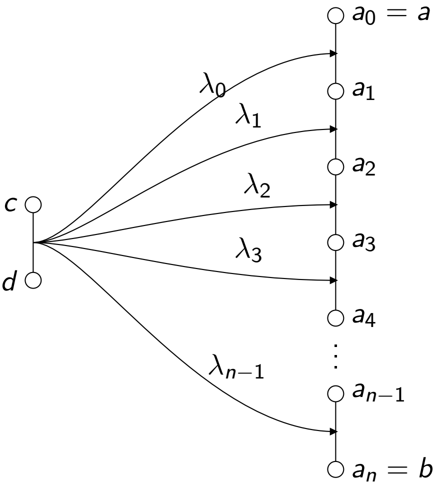

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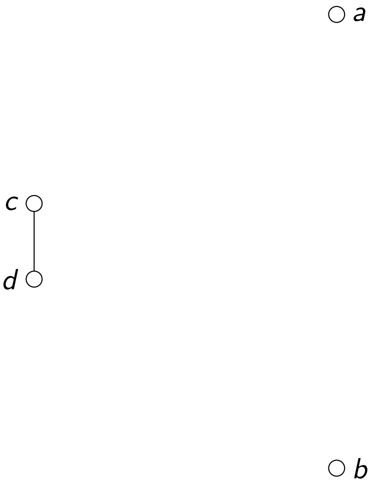

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We write $\{c, d\} \looparrowright^n_k \{a, b\}$
provided
λ_i is a k-translation for each
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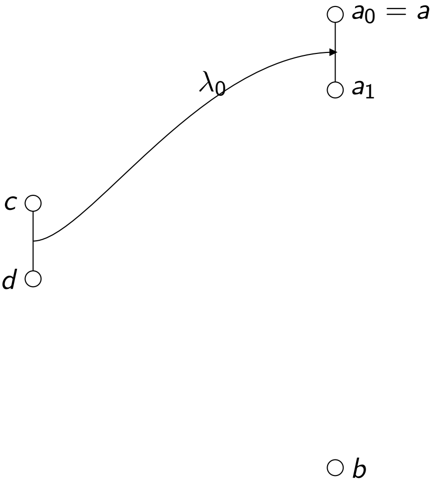

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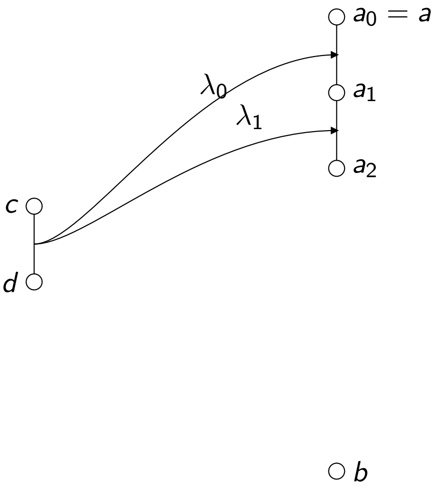

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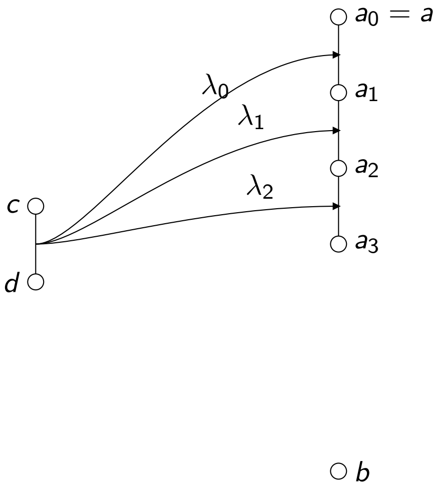

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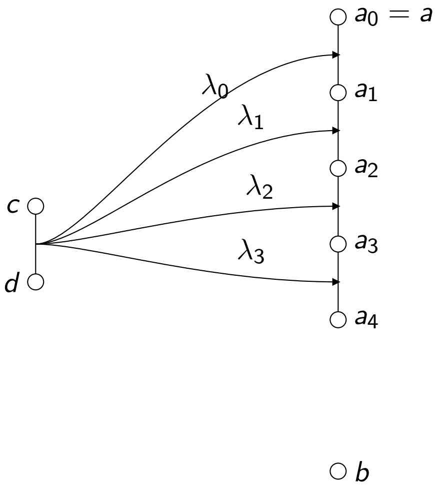

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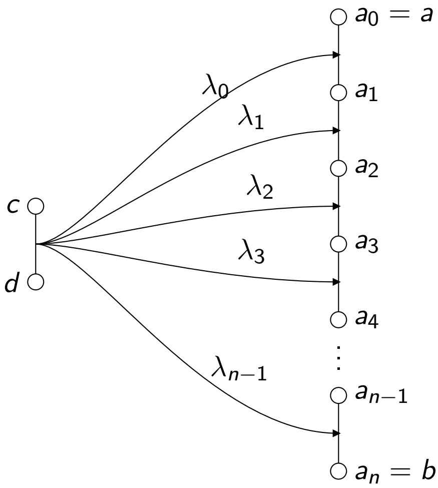
We write $\{c, d\} \looparrowright^n_k \{a, b\}$ provided λ_i is a k-translation for each $i < n$

$\langle a, b \rangle \in \mathrm{Cg}^{\mathbf{A}}(c, d)$ if and only if $\{c, d\} \looparrowright^n_k \{a, b\}$ for some natural numbers k and n

According to Dilworth and Mal'cev



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Definition of Finitely Bounded Critical Depth

Definition

A class $\mathcal{K}$ of algebras of the same finite signature is said to have **finitely bounded critical depth** provided there is a natural number ℓ so that every $\mathbf{A} \in \mathcal{K}$ has a subdirect representation by subdirectly irreducible algebras, such that for each subdirectly irreducible algebra $\mathbf{S}$ in the representation and for all $a, b, c, d \in S$ such that $c \neq d$ and $\langle a, b \rangle$ is a critical pair of $\mathbf{S}$ we have $\{c, d\} \pitchfork\!\!\rightarrow^{n}_{\ell} \{a, b\}$ for some natural number n.

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When a Variety $\mathcal{V}$ Has a Finite Residual Bound We Know

- $\mathcal{V}$ is locally finite.
- $\mathcal{V}_{\mathsf{si}}$ is a finitely axiomatizable elementary class.
- $\mathcal{V}_{\mathsf{si}} = \mathcal{V}_{\mathsf{fsi}}$.
- $\mathcal{V}_{\mathsf{fsi}}$ is a finitely axiomatizable elementary class.
- $\mathcal{V}$ has finitely bounded critical depth.

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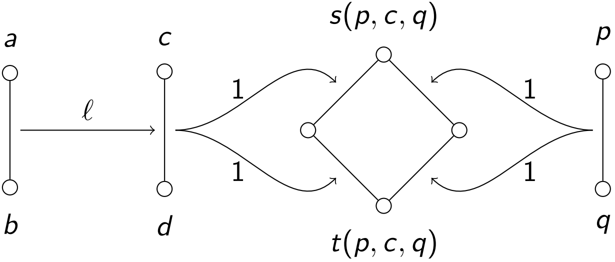
The $\diamondsuit$ Single Sequence Lemma

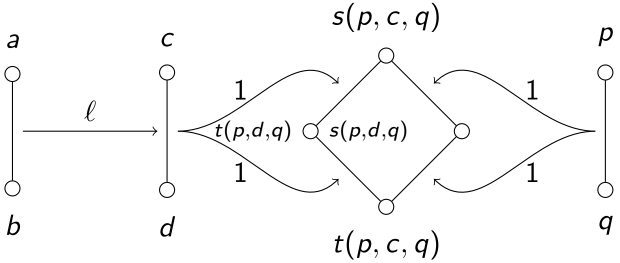
Let $\mathcal{W}$ be a quasivariety of finite signature with Willard terms (treated here as basic operations). Let $\mathcal{U}$ be an axiomatizable class with critical depth bounded by the positive natural number ℓ. For every $\mathbf{A} \in \mathcal{W} \cap \mathsf{S}\,\mathsf{P}\,\mathcal{U}$, and every $(p,q) \in \alpha \subseteq \mathrm{Cg}^{\mathbf{A}}(a,b)$ with $p \neq q$ and α an atom of $\mathrm{Con}\,\mathbf{A}$, there are elements $c, d \in A$ and matched Willard terms $s(x,y,z), t(x,y,z)$ so that

$$\begin{aligned}
\{a,b\} &\looparrowright^{1}_{\ell} \{c,d\}\\
\{c,d\} &\looparrowright^{2}_{1} \{s(p,c,q), t(p,c,q)\}\\
\{p,q\} &\looparrowright^{2}_{1} \{s(p,c,q), t(p,c,q)\}, \text{ and}\\
s(p,c,q) &\neq t(p,c,q).
\end{aligned}$$

The $\diamondsuit$ Single Sequence Lemma



The ♢ Single Sequence Lemma



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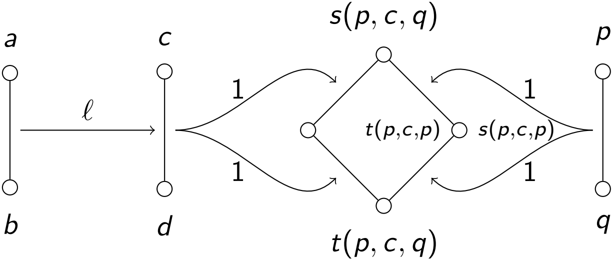


1991 was a very good year. . .

Looking toward quasivarieties

Outline

Victor advanced the foundations

Don Pigozzi widens the field

Pigozzi's Finite Basis Theorem, 1988

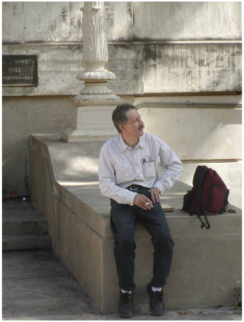
Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ is relatively congruence distributive, and
- $\mathcal{W}$ is finitely generated,

then $\mathcal{W}$ is finitely based.

Outline

Can it all be done again, for quasivarieties?



Wieslaw Dziobiak offers an alternative proof of Pigozzi's Finite Basis Theorem, 1991.

Can it all be done again, for quasivarieties?

A Theorem of Anvar Nurakunov in 1990 and, Janusz Czelakowski and Wieslaw Dziobiak in 1996

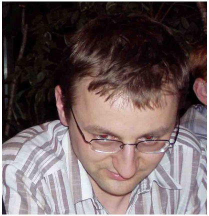
Let $\mathcal{W}$ be a quasivariety of finite signature.

If

- $\mathcal{W}$ has definable relative principal congruence relations, and
- $\mathcal{W}_{\mathrm{rfsi}}$ is finitely axiomatizable,

then $\mathcal{W}$ is finitely based.

Can it all be done again, for quasivarieties?



Theorem of Anvar Nurakunov and Michal Stronkowski, 2009

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has definable relative principal subcongruences, and
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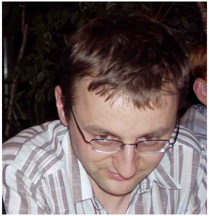

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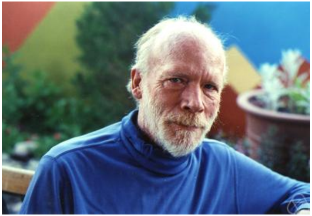
This gives another proof of Pigozzi's Finite Basis Theorem.

Outline

Quasivarieties with pseudocomplemented congruences



Theorem of Miklós Maróti and Ralph McKenzie, 2004

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has pseudocomplemented congruences,
- $\mathcal{U}_n$ is the class of all algebras with no more than n elements, and
- φ is a positive universal sentence,

then $\mathcal{W} \cap \mathrm{S}\,\mathrm{P}\,\mathcal{U}_n \cap \mathrm{S}\,\mathrm{P}\,\mathrm{Mod}\,\varphi$ is finitely axiomatizable relative to $\mathcal{W}$.

Corollaries drawn by Miklós and Ralph

Corollary A

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has pseudocomplemented congruences, and
- $\mathcal{W} \subseteq \mathcal{U}_n$ for some natural number n,

then $\mathcal{W}$ is contained in a finitely axiomatizable locally finite quasivariety.

Corollaries drawn by Miklós and Ralph

Corollary B

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has meet-semidistributive congruences, and
- $\mathcal{W}$ is finitely generated,

then $\mathcal{W}$ is contained in a finitely axiomatizable finitely generated meet-semidistributive quasivariety.

Corollaries drawn by Miklós and Ralph

Corollary C

Let $\mathcal{W}$ be a quasivariety of finte signature.
If

- $\mathcal{W}$ has pseudocomplemented congruences,
- $\mathcal{W}$ is generated by a finite set $\mathcal{K}$ of finite algebras, and
- $\mathsf{H}\,\mathsf{S}\,\mathcal{K} \subseteq \mathcal{W}$,

then $\mathcal{W}$ is finitely axiomatizable.

Willard's Finite Basis Theorem is an application of this corollary.

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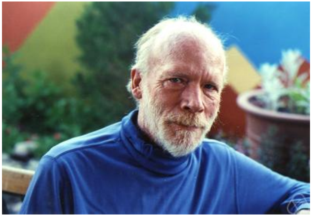
Willard's Finite Basis Theorem is an application of this corollary.

And there is more...

Theorem of Maróti and McKenzie, 2004

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has Willard terms,
- $\mathcal{U}_n$ is the class of all algebras with no more than n elements,
- φ is a universal sentence, and
- there is a quasivariety with the weak extension property between $\mathcal{W} \cap \mathsf{S}\,\mathsf{P}\,\mathcal{U}_n \cap \mathsf{S}\,\mathsf{P}\,\mathrm{Mod}\,\varphi$ and $\mathsf{S}\,\mathsf{P}\,\mathrm{Mod}\,\varphi$,

then $\mathcal{W} \cap \mathsf{S}\,\mathsf{P}\,\mathcal{U}_n \cap \mathsf{S}\,\mathsf{P}\,\mathrm{Mod}\,\varphi$ is finitely axiomatizable relative to $\mathcal{W}$.

Two more corollaries drawn by Miklós and Ralph

Corollary D

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has pseudocomplemented congruences,
- $\mathcal{W}$ has the weak extension property, and
- $\mathcal{W}$ is finitely generated,

then $\mathcal{W}$ is finitely axiomatizable.

Two more corollaries drawn by Miklós and Ralph

Corollary E: Pigozzi's Finite Basis Theorem

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has distributive relative congruences, and
- $\mathcal{W}$ is finitely generated,

then $\mathcal{W}$ is finitely axiomatizable.

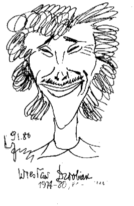
About the weak extension property

Let $\mathcal{W}$ be a quasivariety and $\mathbf{A} \in \mathcal{W}$. Given $\theta \in \operatorname{Con} \mathbf{A}$ it may happen that $\mathbf{A}/\theta \in \mathcal{W}$. In this case, θ is a **relative** congruence. The set of all relative congruences of $\mathbf{A}$ is closed under intersection. But θ might not be a relative congruence. Let $\theta^{\mathcal{W}}$ denote the smallest relative congruence that includes θ.

The quasivariety $\mathcal{W}$ has the **weak extension property** provided for all $\mathbf{A} \in \mathcal{W}$ and all $\theta_0, \theta_1 \in \operatorname{Con} \mathbf{A}$

if $\theta_0 \cap \theta_1$ is trivial, then $\theta_0^{\mathcal{W}} \cap \theta_1^{\mathcal{W}}$ is also trivial.

Getting Ross Willard's Finite Basis Theorem for quasivarieties



Theorem of Wieslaw Dziobiak, 2009

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has meet-semidistributive relative congruence lattices, and
- $\mathcal{W}$ is finitely generated,

then $\mathcal{W}$ is finitely axiomatizable.

Getting Ross Willard's Finite Basis Theorem for quasivarieties

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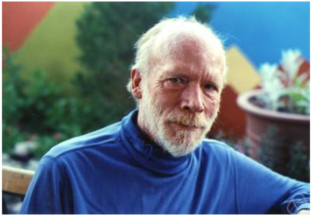

Theorem of Ralph McKenzie, 2009

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Theorem of Anvar Nurakunov, 2009

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Getting Ross Willard's Finite Basis Theorem for quasivarieties

Not the same gang!

Theorem of the Gang of Four 2009

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has meet-semidistributive relative congruence lattices, and
- $\mathcal{W}$ is finitely generated,

then $\mathcal{W}$ is finitely axiomatizable.

A working lemma from the Gang of Four

Another Working Lemma

Let $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{U}$ be quasivarieties of the same finite signature.
If

- $\mathcal{V}$ has the weak extension property, and
- $\mathcal{U}$ is finitely generated,

then $\mathcal{V}$ is finitely based relative to $\mathcal{W}$.

A working lemma from the Gang of Four

Another Working Lemma, version A

Let $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{U}$ be quasivarieties of the same finite signature.
If

- $\mathcal{V}$ has the weak extension property,
- $\mathcal{W}$ is finitely based, and
- $\mathcal{U}$ is finitely generated,

then $\mathcal{V}$ is finitely based.

A working lemma from the Gang of Four

Another Working Lemma, version B

Let $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{U}$ be quasivarieties of the same finite signature.
If

- $\mathcal{V}$ is a variety,
- $\mathcal{W}$ is finitely based, and
- $\mathcal{U}$ is finitely generated,

then $\mathcal{V}$ is finitely based.

A working lemma from the Gang of Four

A consequence of this new working lemma

No nonfinitely based variety of finite signature can be included in any finitely based finitely generated quasivariety.

Following Misha Volkov, we can rephrase this as

> *Every nonfinitely based variety of finite signature is strongly nonfinitely q-based.*

Behind the scenes: Combining ideas of Maróti, McKenzie, Baker, McNulty, and Wang

The Critical Depth Principal Meet Lemma

Let $\mathcal{W}$ be a quasivariety of finite signature that has Willard terms. Let $\mathcal{U}$ be an axiomatizable class with critical depth bounded by the natural number ℓ and let $\mathsf{S}\,\mathsf{P}\,\mathcal{U}$ be locally finite. Then for all natural numbers $m > 0$ the nontriviality of m-fold principal meets is defined by

$$\exists u, v \left[\neg u \approx v \,\&\, \bigwedge_{i<m} \left(\{x_i, y_i\} \looparrowright_{\ell}^{1} \circ \looparrowright_{m}^{2^m} \{u, v\} \right) \right]$$

in the quasivariety $\mathcal{W} \cap \mathsf{S}\,\mathsf{P}\,\mathcal{U}$.

Behind the scenes: Combining ideas of Maróti, McKenzie, Baker, McNulty, and Wang

Theorem of Maróti and McKenzie, upgrade

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has pseudocomplemented congruences,
- $\mathcal{U}$ is a class of algebras with critical depth bounded by some positive natural number,
- $\mathcal{U}$ is axiomatized by a positive universal sentence,
- $\mathsf{S}\,\mathsf{P}\,\mathcal{U}$ is locally finite, and
- φ is a positive universal sentence,

then $\mathcal{W} \cap \mathsf{S}\,\mathsf{P}\,\mathcal{U} \cap \mathsf{S}\,\mathsf{P}\,\mathrm{Mod}\,\varphi$ is finitely axiomatizable relative to $\mathcal{W}$.

Problems!

Are any of the following actually theorems?

Let $\mathcal{V}$ be a variety of finite signature.
If

- $\mathcal{V}$ has a Taylor term, and
- $\mathcal{V}$ has a finite residual bound,

then $\mathcal{V}$ is finitely based.

Problems!

Are any of the following actually theorems?

Let $\mathcal{V}$ be a variety of finite signature.
If

- $\mathcal{V}$ is congruence modular,
- $\mathcal{V}$ is finitely generated, and
- $\mathcal{V}_{\mathsf{si}}$ is finitely based,

then $\mathcal{V}$ is finitely based.

Problems!

Are any of the following actually theorems?

Let $\mathcal{W}$ be a quasivariety of finite signature.
If

- $\mathcal{W}$ has modular relative congruence lattices, and
- $\mathcal{W}$ is finitely generated,

then $\mathcal{W}$ is finitely based.

Problems!

Are any of the following actually theorems?

Let $\mathcal{K} \subseteq W \subseteq \mathsf{S}\,\mathsf{P}\,\mathcal{U}$ be quasivarieties of the same finite signature. If

- $\mathcal{K}$ has the weak extension property,
- $\mathsf{S}\,\mathsf{P}\,\mathcal{U}$ is locally finite,
- $\mathcal{U}$ has finitely bounded critical radius, and
- $\mathcal{U}$ is axiomatizable by a positive universal sentence,

then $\mathcal{K}$ is finitely based relative to $\mathcal{W}$.

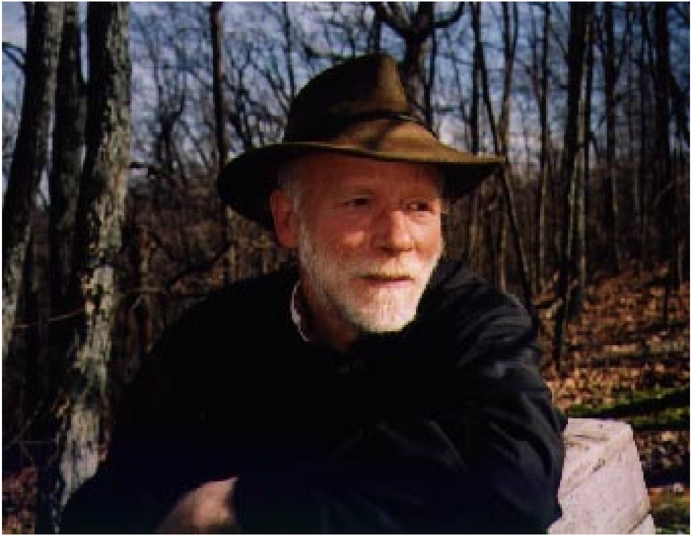
Problems!

Ralph McKenzie's 1970 proof that the variety $\mathcal{V}$ generated by a finite lattice (with finitely many additional operations) is finitely based is syntactical in character. It relies on the construction of a system of clever normal form functions for the varieties $\mathcal{V}^{(n)}$. Can such a syntactic approach succeed in any of the more general settings seen above?

Algebras, Lattices, Varieties 1

Algebras, Lattices, Varieties 2

Where is *that* flute?



Which way is Hamlin?